5-2 Verifying Trigonometric Identities

Verify each identity.

1. \((\sec^2 \theta - 1) \cos^2 \theta = \sin^2 \theta\)

**SOLUTION:**
\[
(\sec^2 \theta - 1) \cos^2 \theta \\
= (\tan^2 \theta) \cos^2 \theta \quad \text{Pythagorean Identity} \\
= \left(\frac{\sin^2 \theta}{\cos^2 \theta}\right) \cos^2 \theta \quad \text{Quotient Identity} \\
= \sin^2 \theta \quad \text{Multiply and divide out common factor.}
\]

2. \(\sec^2 \theta (1 - \cos^2 \theta) = \tan^2 \theta\)

**SOLUTION:**
\[
\sec^2 \theta (1 - \cos^2 \theta) \\
= \sec^2 \theta - \sec^2 \theta \cos^2 \theta \quad \text{Distributive Property} \\
= \sec^2 \theta - \frac{1}{\cos^2 \theta} \cdot \cos^2 \theta \quad \text{Reciprocal Identity} \\
= \sec^2 \theta - 1 \quad \text{Multiply and divide out common factor.} \\
= \tan^2 \theta \quad \text{Pythagorean Identity}
\]

3. \(\sin \theta - \sin \theta \cos^2 \theta = \sin^3 \theta\)

**SOLUTION:**
\[
\sin \theta - \sin \theta \cos^2 \theta \\
= \sin \theta (1 - \cos^2 \theta) \quad \text{Factor.} \\
= \sin \theta \sin^2 \theta \quad \text{Pythagorean Identity} \\
= \sin^3 \theta \quad \text{Multiply.}
\]

4. \(\csc \theta - \cos \theta \cot \theta = \sin \theta\)

**SOLUTION:**
\[
\csc \theta - \cos \theta \cot \theta \\
= \frac{1}{\sin \theta} - \cos \theta \left(\frac{\cos \theta}{\sin \theta}\right) \quad \text{Reciprocal and Quotient Identities} \\
= \frac{1 - \cos^2 \theta}{\sin \theta} \quad \text{Write as a fraction with a common denominator.} \\
= \frac{\sin^2 \theta}{\sin \theta} \quad \text{Pythagorean Identity} \\
= \sin \theta \quad \text{Divide out common factor of \(\sin \theta\).}
\]

5-2 Verifying Trigonometric Identities

5. \( \cot^2 \theta \csc^2 \theta - \cot^2 \theta = \cot^4 \theta \)

**SOLUTION:**
\[
\cot^2 \theta \csc^2 \theta - \cot^2 \theta \\
= \cot^2 \theta (\csc^2 \theta - 1) \quad \text{Factor.} \\
= \cot^2 \theta \cot^2 \theta \quad \text{Pythagorean Identity} \\
= \cot^4 \theta \quad \text{Multiply and add exponents.}
\]

6. \( \tan \theta \csc^2 \theta - \tan \theta = \cot \theta \)

**SOLUTION:**
\[
\tan \theta \csc^2 \theta - \tan \theta \\
= \tan \theta (\csc^2 \theta - 1) \quad \text{Factor} \\
= \tan \theta \cot^2 \theta \quad \text{Pythagorean Identity} \\
= \sin \theta \cos^2 \theta \quad \text{Quotient Identities} \\
= \cos \theta \quad \text{Multiply and divide common factors.} \\
= \cot \theta \quad \text{Quotient Identity}
\]

7. \( \sec \theta - \frac{\sin \theta}{\cos \theta} = \cot \theta \)

**SOLUTION:**
\[
\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \\
= \frac{1}{\sin \theta} \quad \text{Reciprocal Identity} \\
= \frac{\cos \theta}{\sin \theta} \quad \text{Common denominator} \\
= \frac{\sin \theta - \sin^2 \theta}{\sin \theta \cos \theta} \quad \text{Write as a fraction with a common denominator.} \\
= \frac{\cos^2 \theta}{\sin \theta \cos \theta} \quad \text{Pythagorean Identity} \\
= \frac{\cos \theta}{\sin \theta} \quad \text{Divide out common factor of \( \cos \theta \).} \\
= \cot \theta \quad \text{Quotient Identity}
\]
5-2 Verifying Trigonometric Identities

8. \[ \frac{\sin \theta + \cos \theta}{\sin \theta \cos \theta} = 2 \csc \theta \]

**SOLUTION:**

\[
\frac{\sin \theta + 1 - \cos \theta}{1 - \cos \theta + \sin \theta}
\]

Rewrite 1 using the common denominator.

\[
\frac{\sin \theta \cdot \sin \theta + 1 - \cos \theta \cdot \sin \theta}{\sin \theta \cdot \sin \theta + 1 - \cos \theta \cdot \sin \theta}
\]

Multiply.

\[
\frac{\sin^2 \theta + 1 - 2 \cos \theta}{\sin \theta(1 - \cos \theta)}
\]

Write as a fraction with a common denominator.

Pythagorean Identity

\[
\frac{2}{\sin \theta}
\]

Add.

Factor.

\[
2 \sec \theta
\]

Divide out common factor of \((1 - \cos \theta)\).

Reciprocal Identity

9. \[ \frac{\cos \theta}{1 + \sin \theta} + \tan \theta = \sec \theta \]

**SOLUTION:**

\[
\frac{\cos \theta + \sin \theta}{1 + \sin \theta}
\]

Quotient Identity

\[
\frac{\cos \theta \cdot \cos \theta + \sin \theta \cdot \sin \theta}{\cos \theta \cdot (1 + \sin \theta)}
\]

Rewrite 1 using the common denominator.

\[
\frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta(1 + \sin \theta)}
\]

Multiply.

Pythagorean Identity

\[
\frac{1}{\cos \theta(1 + \sin \theta)}
\]

Write as a fraction with a common denominator.

Divide out common factor of \((1 + \sin \theta)\).

Reciprocal Identity
5-2 Verifying Trigonometric Identities

10. \(\frac{\sin\theta}{1 + \cot\theta} + \frac{\cos\theta}{1 + \tan\theta} = \sin\theta + \cos\theta\)

**SOLUTION:**

\[
\begin{align*}
\frac{\sin\theta}{1 - \cot\theta} + \frac{\cos\theta}{1 - \tan\theta} &= \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\cos\theta}{\frac{\sin\theta}{\cos\theta}} \\
&= \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\cos\theta} \\
&= \frac{\sin\theta - \cos\theta}{\cos\theta - \sin\theta} \\
&= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta - \cos\theta} \\
&= \frac{(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)}{\sin\theta - \cos\theta} \\
&= \sin\theta + \cos\theta
\end{align*}
\]

Quotient Identity

Rewrite 1 using the common denominator.

Write denominators as fractions with common denominators.

Simplify fractions.

Factor out -1.

Write as a fraction with common denominator.

Factor numerator.

Divide out common factor of \((\sin\theta - \cos\theta)\).
5-2 Verifying Trigonometric Identities

11. \( \frac{1}{\tan 2\theta} + \frac{1}{\cot 2\theta} = 1 \)

**SOLUTION:**

\[
\begin{align*}
\frac{1}{\tan^2 \theta} + \frac{1}{\cot^2 \theta} &= \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \\
&= \frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} \\
&= \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta} \\
&= 1
\end{align*}
\]

Quotient Identity

Rewrite 1 using the common denominator.

Write denominators as fractions with common denominators.

Simplify fractions.

Factor out -1.

Common denominator

Write as a fraction with a common denominator.

Divide out common factor of \((\cos^2 \theta - \sin^2 \theta)\).
12. \[
\frac{1}{\csc \theta + 1} + \frac{1}{\csc \theta - 1} = 2 \sec^2 \theta \sin \theta
\]

**SOLUTION:**

\[
\begin{align*}
\frac{1}{\csc \theta + 1} + \frac{1}{\csc \theta - 1} &= \frac{\csc \theta - 1}{\csc \theta - 1} + \frac{\csc \theta + 1}{\csc \theta + 1} - \frac{1}{\csc \theta - 1} - \frac{1}{\csc \theta + 1} \\
&= \frac{\csc \theta - 1}{\csc \theta - 1} + \frac{\csc \theta + 1}{\csc \theta + 1} - \frac{1}{\csc \theta - 1} - \frac{1}{\csc \theta + 1} \\
&= \frac{\csc \theta - 1}{\csc \theta - 1} + \frac{\csc \theta + 1}{\csc \theta + 1} - \frac{1}{\csc \theta - 1} - \frac{1}{\csc \theta + 1} \\
&= \frac{2\csc \theta}{\csc^2 \theta - 1} \\
&= \frac{2\csc \theta}{\csc \theta \cot \theta} \\
&= \frac{2\left(\frac{1}{\sin \theta}\right)}{\cos^2 \theta} \\
&= \frac{2\sin \theta}{\sin \theta \cos^2 \theta} \\
&= \frac{\cos \theta}{\cos^2 \theta} \\
&= \left(\frac{2}{\cos \theta}\right) \sin \theta \\
&= 2 \sec^2 \theta \sin \theta
\end{align*}
\]

Common denominator

Multiply.

Write as a fraction with a common denominator.

Pythagorean Identity

Reciprocal and Quotient Identities

Multiply by the reciprocal of the denominator.

Multiply.

Factor.

Reciprocal Identity

13. \[(\csc \theta - \cot \theta)(\csc \theta + \cot \theta) = 1\]

**SOLUTION:**

\[(\csc \theta - \cot \theta)(\csc \theta + \cot \theta)\]

\[= \csc^2 \theta - \cot^2 \theta \]

Multiply.

Pythagorean Identity

14. \[
\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta
\]

**SOLUTION:**

\[
\begin{align*}
\cos^4 \theta - \sin^4 \theta &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\
&= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\
&= 1(\cos^2 \theta - \sin^2 \theta) \\
&= \cos^2 \theta - \sin^2 \theta
\end{align*}
\]

Factor.

Pythagorean Identity

Multiply.
Verify each identity.

1. \( \sec^2 \theta - 1 \) \( \cos \) \( \theta \) = \( \sin \) \( \theta \)

**SOLUTION:**

\[
\frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta}
\]

Common denominator

\[
= \frac{1}{1 - \sin \theta} \cdot \frac{1 + \sin \theta}{1 + \sin \theta} + \frac{1}{1 + \sin \theta} \cdot \frac{1 - \sin \theta}{1 - \sin \theta}
\]

Multiply.

\[
= \frac{1 + \sin \theta}{1 - \sin \theta} + \frac{1 - \sin \theta}{1 - \sin \theta}
\]

Write as a fraction with a common denominator.

\[
= \frac{2}{\cos^2 \theta}
\]

Pythagorean Identity

\[
= 2 \sec^2 \theta
\]

Reciprocal Identity

16. \( \frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta \)

**SOLUTION:**

\[
\frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta}
\]

Common denominator

\[
= \frac{\cos \theta(1 - \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)} + \frac{\cos \theta(1 + \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)}
\]

Multiply.

\[
= \frac{\cos \theta(1 - \sin \theta) + \cos \theta(1 + \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)}
\]

Write as a fraction with a common denominator.

\[
= \frac{\cos \theta - \sin \theta \cos \theta + \cos \theta + \sin \theta \cos \theta}{1 - \sin^2 \theta}
\]

Multiply.

\[
= \frac{2 \cos \theta}{1 - \sin^2 \theta}
\]

Simplify the numerator.

\[
= \frac{2 \cos \theta}{\cos^2 \theta}
\]

Pythagorean Identity

\[
= \frac{2}{\cos \theta}
\]

Divide out common factor of \( \cos \theta \).

\[
= 2 \sec \theta
\]

Quotient Identity
5-2 Verifying Trigonometric Identities

17. \( \csc^4 \theta - \cot^4 \theta = 2 \cot^2 \theta + 1 \)

**SOLUTION:**

\[
\begin{align*}
\csc^4 \theta - \cot^4 \theta &= (\csc^2 \theta - \cot^2 \theta)(\csc^2 \theta + \cot^2 \theta) \\
&= [\csc^2 \theta - (\csc^2 \theta - 1)][\csc^2 \theta + (\csc^2 \theta - 1)] \\
&= [\csc^2 \theta - \csc^2 \theta + 1][\csc^2 \theta + \csc^2 \theta - 1] \\
&= 1[2\csc^2 \theta - 1] \\
&= 2\csc^2 \theta - 1 \\
&= 2(\cot^2 \theta + 1) - 1 \\
&= 2\cot^2 \theta + 2 - 1 \\
&= 2\cot^2 \theta + 1
\end{align*}
\]

18. \( \frac{\csc 2\theta + 2 \csc \theta + 3}{\csc 2\theta + 1} = \frac{\csc \theta + 3}{\csc \theta + 1} \)

**SOLUTION:**

\[
\begin{align*}
\frac{\csc^2 \theta + 2 \csc \theta - 3}{\csc^2 \theta - 1} &= \frac{(\csc \theta + 3)(\csc \theta - 1)}{(\csc \theta + 1)(\csc \theta - 1)} \\
&= \frac{\csc \theta + 3}{\csc \theta + 1}
\end{align*}
\]

Factor the numerator and denominator.

Divide out common factor of \((\csc \theta - 1)\).
5-2 Verifying Trigonometric Identities

19. **FIREWORKS** If a rocket is launched from ground level, the maximum height that it reaches is given by \( h = \frac{v^2 \sin 2\theta}{2g} \), where \( \theta \) is the angle between the ground and the initial path of the rocket, \( v \) is the rocket’s initial speed, and \( g \) is the acceleration due to gravity, 9.8 meters per second squared.

![Rocket Launch Diagram]

a. Verify that \( \frac{v^2 \sin 2\theta}{2g} = \frac{v^2 \tan 2\theta}{2g \sec 2\theta} \).

b. Suppose a second rocket is fired at an angle of 80° from the ground with an initial speed of 110 meters per second. Find the maximum height of the rocket.

**SOLUTION:**

a. 

\[
\frac{v^2 \tan^2 \theta}{2g \sec^2 \theta} = \frac{v^2 \left( \frac{\sin^2 \theta}{\cos^2 \theta} \right)}{2g \left( \frac{1}{\cos^2 \theta} \right)}
\]

Quotient and Reciprocal Identities

\[
= \frac{v^2 \sin^2 \theta}{2g}
\]

Divide out the common factor \( \frac{1}{\cos^2 \theta} \).

b. Evaluate the expression \( \frac{v^2 \sin^2 \theta}{2g} \) for \( v = 110 \) m, \( \theta = 80^\circ \), and \( g = 9.8 \) m/s².

\[
\frac{v^2 \sin^2 \theta}{2g} = \frac{110^2 \sin^2 80^\circ}{2(9.8)}
\]

\[ \approx 598.7 \]

The maximum height of the rocket is about 598.7 meters.
5-2 Verifying Trigonometric Identities

Verify each identity.

20. (csc $\theta + \cot \theta)(1 - \cos \theta) = \sin \theta$

**SOLUTION:**

\[
(csc \theta + \cot \theta)(1 - \cos \theta) \\
= \csc \theta - \csc \theta \cos \theta + \cot \theta - \cot \theta \cos \theta \quad \text{Multiply binomials.} \\
= \frac{1}{\sin \theta} - \left( \frac{1}{\sin \theta} \right) \cos \theta + \frac{\cos \theta}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \cos \theta \quad \text{Reciprocal and Quotient Identities} \\
= \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \cos \theta \\
= 1 - \cos \theta + \cos \theta - \cos^2 \theta \quad \text{Multiply.} \\
= \frac{1 - \cos^2 \theta}{\sin \theta} \quad \text{Write as one fraction with a common denominator.} \\
= \frac{\sin^2 \theta}{\sin \theta} \quad \text{Simplify numerator.} \\
= \frac{\sin \theta}{\sin \theta} \quad \text{Pythagorean Identity} \\
= \sin \theta \quad \text{Divide out common factor.}
\]

21. $\sin^2 \theta \tan^2 \theta = \tan^2 \theta - \sin^2 \theta$

**SOLUTION:**

\[
\tan^2 \theta - \sin^2 \theta \\
= \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \quad \text{Quotient Identity} \\
= \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta \cdot 1 \\
= \frac{\sin^2 \theta}{\cos^2 \theta} - (\sin^2 \theta) \left( \frac{\cos^2 \theta}{\cos^2 \theta} \right) \quad \text{Rewrite 1 using the common denominator.} \\
= \frac{\sin^2 \theta}{\cos^2 \theta} - \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \\
= \frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \quad \text{Multiply.} \\
= \frac{\sin^2 \theta(1 - \cos^2 \theta)}{\cos^2 \theta} \quad \text{Write as a fraction with a common denominator.} \\
= \frac{\sin^2 \theta \sin^2 \theta}{\cos^2 \theta} \quad \text{Factor the numerator.} \\
= \frac{\sin^2 \theta}{\cos^2 \theta} \quad \text{Pythagorean Identity} \\
= \sin^2 \theta \left( \frac{\sin^2 \theta}{\cos^2 \theta} \right) \quad \text{Factor.} \\
= \sin^2 \theta \tan^2 \theta \quad \text{Quotient Identity}
\]
Verify each identity.

1. \[ \frac{\tan 2\theta}{\cos 2\theta} = \frac{\cos 2\theta}{\sin 2\theta} \]

\[ \text{SOLUTION:} \]

Reciprocal Identities

2. \[ \frac{1}{\cot 2\theta} = \frac{\cos 2\theta}{\sin 2\theta} \]

\[ \text{SOLUTION:} \]

Rewrite 1 using the common denominator.

3. \[ \sin \theta = \cos \theta \]

Which of the following is not equivalent to \( \cos \theta \), when \( 0 < \theta < \pi \)?

A  
B  
C  
D  

\[ \text{SOLUTION:} \]

Pythagorean Identity

4. \[ \sin^2 \theta + \cos^2 \theta = 1 \]

Simplify each expression.

5. \[ \tan x \]

6. \[ \sin x \]

7. \[ \cos x \]

8. \[ \sec x \]

9. \[ \csc x \]

10. \[ \cot x \]

11. \[ \sec \alpha \]

12. \[ \csc \alpha \]

13. \[ \cot \alpha \]

14. \[ \sin \alpha \]

15. \[ \cos \alpha \]

16. \[ \sec \beta \]

17. \[ \csc \beta \]

18. \[ \cot \beta \]

19. \[ \sin \beta \]

20. \[ \cos \beta \]

21. \[ \sec \gamma \]

22. \[ \csc \gamma \]

23. \[ \cot \gamma \]

24. \[ \sin \gamma \]

25. \[ \cos \gamma \]

26. \[ \sec \delta \]

27. \[ \csc \delta \]

28. \[ \cot \delta \]

29. \[ \sin \delta \]

30. \[ \cos \delta \]

31. \[ \sec \theta \]

32. \[ \csc \theta \]

33. \[ \cot \theta \]

34. \[ \sin \theta \]

35. \[ \cos \theta \]

36. \[ \sec \phi \]

37. \[ \csc \phi \]

38. \[ \cot \phi \]

39. \[ \sin \phi \]

40. \[ \cos \phi \]

41. \[ \sec \psi \]

42. \[ \csc \psi \]

43. \[ \cot \psi \]

44. \[ \sin \psi \]

45. \[ \cos \psi \]

46. \[ \sec \chi \]

47. \[ \csc \chi \]

48. \[ \cot \chi \]

49. \[ \sin \chi \]

50. \[ \cos \chi \]

51. \[ \sec \xi \]

52. \[ \csc \xi \]

53. \[ \cot \xi \]

54. \[ \sin \xi \]

55. \[ \cos \xi \]

56. \[ \sec \psi \]

57. \[ \csc \psi \]

58. \[ \cot \psi \]

59. \[ \sin \psi \]

60. \[ \cos \psi \]

61. \[ \sec \phi \]

62. \[ \csc \phi \]

63. \[ \cot \phi \]

64. \[ \sin \phi \]

65. \[ \cos \phi \]

66. \[ \sec \theta \]

67. \[ \csc \theta \]

68. \[ \cot \theta \]

69. \[ \sin \theta \]

70. \[ \cos \theta \]

71. \[ \sec \alpha \]

72. \[ \csc \alpha \]

73. \[ \cot \alpha \]

74. \[ \sin \alpha \]

75. \[ \cos \alpha \]

76. \[ \sec \beta \]

77. \[ \csc \beta \]

78. \[ \cot \beta \]

79. \[ \sin \beta \]

80. \[ \cos \beta \]

81. \[ \sec \gamma \]

82. \[ \csc \gamma \]

83. \[ \cot \gamma \]

84. \[ \sin \gamma \]

85. \[ \cos \gamma \]

86. \[ \sec \delta \]

87. \[ \csc \delta \]

88. \[ \cot \delta \]

89. \[ \sin \delta \]

90. \[ \cos \delta \]

91. \[ \sec \theta \]

92. \[ \csc \theta \]

93. \[ \cot \theta \]

94. \[ \sin \theta \]

95. \[ \cos \theta \]

96. \[ \sec \phi \]

97. \[ \csc \phi \]

98. \[ \cot \phi \]

99. \[ \sin \phi \]

100. \[ \cos \phi \]

101. \[ \sec \psi \]

102. \[ \csc \psi \]

103. \[ \cot \psi \]

104. \[ \sin \psi \]

105. \[ \cos \psi \]

106. \[ \sec \chi \]

107. \[ \csc \chi \]

108. \[ \cot \chi \]

109. \[ \sin \chi \]

110. \[ \cos \chi \]

111. \[ \sec \xi \]

112. \[ \csc \xi \]

113. \[ \cot \xi \]

114. \[ \sin \xi \]

115. \[ \cos \xi \]

116. \[ \sec \psi \]

117. \[ \csc \psi \]

118. \[ \cot \psi \]

119. \[ \sin \psi \]

120. \[ \cos \psi \]

121. \[ \sec \phi \]

122. \[ \csc \phi \]

123. \[ \cot \phi \]

124. \[ \sin \phi \]

125. \[ \cos \phi \]

126. \[ \sec \theta \]

127. \[ \csc \theta \]

128. \[ \cot \theta \]

129. \[ \sin \theta \]

130. \[ \cos \theta \]

131. \[ \sec \alpha \]

132. \[ \csc \alpha \]

133. \[ \cot \alpha \]

134. \[ \sin \alpha \]

135. \[ \cos \alpha \]

136. \[ \sec \beta \]

137. \[ \csc \beta \]

138. \[ \cot \beta \]

139. \[ \sin \beta \]

140. \[ \cos \beta \]

141. \[ \sec \gamma \]

142. \[ \csc \gamma \]

143. \[ \cot \gamma \]

144. \[ \sin \gamma \]

145. \[ \cos \gamma \]

146. \[ \sec \delta \]

147. \[ \csc \delta \]

148. \[ \cot \delta \]

149. \[ \sin \delta \]

150. \[ \cos \delta \]

151. \[ \sec \theta \]

152. \[ \csc \theta \]

153. \[ \cot \theta \]

154. \[ \sin \theta \]

155. \[ \cos \theta \]

156. \[ \sec \phi \]

157. \[ \csc \phi \]

158. \[ \cot \phi \]

159. \[ \sin \phi \]

160. \[ \cos \phi \]

161. \[ \sec \psi \]

162. \[ \csc \psi \]

163. \[ \cot \psi \]

164. \[ \sin \psi \]

165. \[ \cos \psi \]

166. \[ \sec \chi \]

167. \[ \csc \chi \]

168. \[ \cot \chi \]

169. \[ \sin \chi \]

170. \[ \cos \chi \]

171. \[ \sec \xi \]

172. \[ \csc \xi \]

173. \[ \cot \xi \]

174. \[ \sin \xi \]

175. \[ \cos \xi \]

176. \[ \sec \psi \]

177. \[ \csc \psi \]

178. \[ \cot \psi \]

179. \[ \sin \psi \]

180. \[ \cos \psi \]

181. \[ \sec \phi \]

182. \[ \csc \phi \]

183. \[ \cot \phi \]

184. \[ \sin \phi \]

185. \[ \cos \phi \]

186. \[ \sec \theta \]

187. \[ \csc \theta \]

188. \[ \cot \theta \]

189. \[ \sin \theta \]

190. \[ \cos \theta \]
23. \(\frac{1 + \csc \theta}{\sec \theta} = \cos \theta + \cot \theta\)

**SOLUTION:**

\[
\begin{align*}
\frac{1 + \csc \theta}{\sec \theta} &= \\
&= \frac{1}{\sec \theta} + \frac{1}{\csc \theta} \quad \text{Rewrite 1 using the common denominator.} \\
&= \frac{\sin \theta}{1} + \frac{\cos \theta}{\sin \theta} \quad \text{Rewrite the numerator as a fraction with a common denominator.} \\
&= \frac{\sin \theta \cos \theta + \cos \theta}{\sin \theta} \quad \text{Multiply by the reciprocal of the denominator.} \\
&= \frac{\sin \theta \cos \theta}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \quad \text{Write as a fraction with a common denominator.} \\
&= \cos \theta + \cot \theta \quad \text{Write two fractions.} \\
&= \cos \theta + \frac{\cos \theta}{\sin \theta} \quad \text{Divide out the common factor of } \sin \theta. \\
&= \cos \theta + \cot \theta \quad \text{Quotient Identity}
\end{align*}
\]
5-2 Verifying Trigonometric Identities

24. \((\csc \theta - \cot \theta)^2 = \frac{1-\cos \theta}{1+\cos \theta}\)

**SOLUTION:**

\[
(\csc \theta - \cot \theta)^2 \\
= (\csc \theta - \cot \theta)(\csc \theta - \cot \theta) \\
= \csc^2 \theta - 2\csc \theta \cot \theta + \cot^2 \theta \\
= \frac{1}{\sin^2 \theta} - \frac{2\cos \theta}{\sin \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} \\
= \frac{1}{\sin^2 \theta} - \frac{2\cos \theta \sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta \sin^2 \theta}{\sin^2 \theta} \\
= \frac{1-2\cos \theta + \cos^2 \theta}{\sin^2 \theta} \\
= \frac{1-2\cos \theta + \cos^2 \theta}{1-\cos^2 \theta} \\
= \frac{(1-\cos \theta)^2}{(1+\cos \theta)(1-\cos \theta)} \\
= \frac{1-\cos \theta}{1+\cos \theta}
\]

Rewrite as a product of two binomials.
Multiply binomials.
Reciprocal and Quotient Identities
Multiply fractions.
Write as a fraction with a common denominator.
Pythagorean Identity
Factor the numerator and the denominator.
Divide out the common factor of \((1-\cos \theta)\).
25. \( \frac{1 + \tan 2\theta}{1 - \tan 2\theta} = \frac{1}{2\cos 2\theta} \)

**SOLUTION:**

\[
\begin{align*}
1 + \tan^2 \theta &= \frac{1 + \sin^2 \theta}{\cos^2 \theta} \\
1 - \tan^2 \theta &= \frac{1 - \sin^2 \theta}{\cos^2 \theta} \\
&= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} \\
&= \frac{\cos 2\theta}{\cos^2 \theta - \sin^2 \theta} \\
&= \frac{\cos 2\theta}{\cos^2 \theta - (1 - \cos^2 \theta)} \\
&= \frac{1}{\cos^2 \theta + \cos^2 \theta} \\
&= \frac{1}{2\cos^2 \theta - 1}
\end{align*}
\]

Quotient Identity

Rewrite 1 using the common denominator.

Write the numerator and the denominator as a fraction with a common denominator.

Multiply by the reciprocal of the denominator.

Multiply.

Divide out the common factor of \( \cos^2 \theta \).

Pythagorean Identity

Pythagorean Identity

Multiply in the denominator.

Simplify the denominator.
5-2 Verifying Trigonometric Identities

26. \( \tan^2 \theta \cos^2 \theta = 1 - \cos^2 \theta \)

**SOLUTION:**
\[
\tan^2 \theta \cos^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \cos^2 \theta = \sin^2 \theta \quad \text{Quotient Identity}
\]
\[
= \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \quad \text{Multiply.}
\]
\[
= \sin^2 \theta \quad \text{Divide out common factor of } \cos^2 \theta.
\]
\[
= 1 - \cos^2 \theta \quad \text{Pythagorean Identity}
\]

27. \( \sec \theta - \cos \theta = \tan \theta \sin \theta \)

**SOLUTION:**
\[
\sec \theta - \cos \theta = \frac{1}{\cos \theta} - \cos \theta = \frac{1 - \cos^2 \theta}{\cos \theta}\quad \text{Reciprocal Identity}
\]
\[
= \frac{\sin^2 \theta}{\cos \theta} \quad \text{Rewrite } \cos \theta \text{ using the common denominator.}
\]
\[
= \frac{\sin \theta \cdot \sin \theta}{\cos \theta} \quad \text{Add fractions.}
\]
\[
= \frac{\sin \theta}{\cos \theta} \quad \text{Pythagorean Identity}
\]
\[
= \tan \theta \sin \theta \quad \text{Rewrite as a product of two fractions.}
\]
\[
= \tan \theta \sin \theta \quad \text{Quotient Identity}
\]

28. \( 1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta \)

**SOLUTION:**
\[
1 - \tan^4 \theta = (1 - \tan^2 \theta)(1 + \tan^2 \theta) \quad \text{Factor difference of two squares.}
\]
\[
= [1 - (\sec^2 \theta - 1)](\sec^2 \theta) \quad \text{Pythagorean Identities}
\]
\[
= [1 - \sec^2 \theta + 1](\sec^2 \theta) \quad \text{Distributive Property}
\]
\[
= (2 - \sec^2 \theta)(\sec^2 \theta) \quad \text{Simplify.}
\]
\[
= 2 \sec^2 \theta - \sec^4 \theta \quad \text{Distributive Property}
\]
5-2 Verifying Trigonometric Identities

29. 

\[(\csc \theta - \cot \theta)^2 = \frac{1}{1 + \cos \theta}\]

**SOLUTION:**

\[\begin{align*}
(\csc \theta - \cot \theta)^2 & \quad \text{Reciprocal and Quotient Identities} \\
= \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \right)^2 & \quad \text{Add fractions.} \\
= \left( \frac{1 - \cos \theta}{\sin \theta} \right)^2 & \quad \text{Power of a Quotient} \\
= \frac{(1 - \cos \theta)^2}{\sin^2 \theta} & \quad \text{Pythagorean Identity} \\
= \frac{1 - \cos \theta}{1 + \cos \theta} & \quad \text{Factor.} \\
= \frac{1}{\csc \theta} & \quad \text{Divide out common factor.}
\end{align*}\]

30. 

\[\frac{1 + \tan \theta}{\sin \theta + \cos \theta} = \sec \theta\]

**SOLUTION:**

\[\begin{align*}
\frac{1 + \tan \theta}{\sin \theta + \cos \theta} & \quad \text{Quotient Identity} \\
= \frac{1 + \sin \theta}{\cos \theta} \frac{1}{\sin \theta + \cos \theta} & \quad \text{Rewrite 1 using the common denominator.} \\
= \frac{\cos \theta}{\cos \theta + \sin \theta} \frac{1}{\cos \theta + \sin \theta} & \quad \text{Write the numerator as a fraction with a common denominator.} \\
= \frac{\cos \theta}{\cos \theta + \sin \theta} \frac{1}{\cos \theta + \sin \theta} & \quad \text{Multiply by the reciprocal of the denominator.} \\
= \frac{1}{\cos \theta} & \quad \text{Multiply.} \\
= \sec \theta & \quad \text{Reciprocal Identity}
\end{align*}\]
5-2 Verifying Trigonometric Identities

31. \[ \frac{2 + \csc \theta \sec \theta}{\csc \theta \sec \theta} = (\sin \theta + \cos \theta)^2 \]

**SOLUTION:**
\[
\frac{2 + \csc \theta \sec \theta}{\csc \theta \sec \theta} = \frac{2}{\csc \theta \sec \theta} + \frac{\csc \theta \sec \theta}{\csc \theta \sec \theta} = \frac{2}{\csc \theta \sec \theta} + 1 = 1 \]
\[
= 2 \sin \theta \cos \theta + (\sin^2 \theta + \cos^2 \theta) \quad \text{Reciprocal and Pythagorean Identities} \\
= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \quad \text{Commutative Property of Addition} \\
= (\sin \theta + \cos \theta)^2 \quad \text{Factor Perfect Square Trinomial.}
\]

32. **OPTICS** If two prisms of the same power are placed next to each other, their total power can be determined using \( z = 2p \cos \theta \), where \( z \) is the combined power of the prisms, \( p \) is the power of the individual prisms, and \( \theta \) is the angle between the two prisms. Verify that \( 2p \cos \theta = 2p (1 - \sin^2 \theta) \sec \theta \).

**SOLUTION:**
\[
2p(1 - \sin^2 \theta) \sec \theta = 2p \cos^2 \theta \sec \theta \quad \text{Pythagorean Identity} \\
= 2p \cos^2 \theta \cdot \frac{1}{\cos \theta} \quad \text{Reciprocal Identity} \\
= 2p \cos \theta \quad \text{Divide out the common factor \( \cos \theta \).}
\]

33. **PHOTOGRAPHY** The amount of light passing through a polarization filter can be modeled using \( I = I_m \cos^2 \theta \), where \( I \) is the amount of light passing through the filter, \( I_m \) is the amount of light shined on the filter, and \( \theta \) is the angle of rotation between the light source and the filter. Verify that \( I_m \cos^2 \theta = I_m - \frac{I_m}{\cot 2\theta + 1} \).

**SOLUTION:**
\[
I_m - \frac{I_m}{\cot^2 \theta + 1} = I_m \left( 1 - \frac{1}{\cot^2 \theta + 1} \right) \quad \text{Factor.} \\
= I_m \left( 1 - \frac{1}{\csc^2 \theta} \right) \quad \text{Pythagorean Identity} \\
= I_m (1 - \sin^2 \theta) \quad \text{Reciprocal Identity} \\
= I_m \cos^2 \theta \quad \text{Pythagorean Identity}
\]
5-2 Verifying Trigonometric Identities

**GRAPHING CALCULATOR** Test whether each equation is an identity by graphing. If it appears to be an identity, verify it. If not, find an x-value for which both sides are defined but not equal.

34. \( \frac{\tan x + 1}{\tan x - 1} = \frac{1 + \cot x}{1 - \cot x} \)

**SOLUTION:**

Graph \( Y_1 = \frac{\tan x + 1}{\tan x - 1} \) and then graph \( Y_2 = \frac{1 + \cot x}{1 - \cot x} \).

The graphs of the related functions do not coincide for all values of \( x \) for which both functions are defined. Using the CALC feature on the graphing calculator to find that when \( x = \pi \), \( Y_1 = -1 \) and \( Y_2 \) is undefined. Therefore, the equation is not an identity.
5-2 Verifying Trigonometric Identities

35. \( \sec x + \tan x = \frac{1}{\sec x \tan x} \)

**SOLUTION:**

Graph \( Y_1 = \sec x + \tan x \) and then graph \( Y_2 = \frac{1}{\sec x - \tan x} \).

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

\[
\frac{1}{\sec x - \tan x} = \frac{1}{\sec x - \tan x}
\]

**Reciprocal and Quotient Identities**

\[
\frac{1}{\sec x - \tan x} = \frac{1}{\sin x} = \frac{1}{\cos x}
\]

Subtract fractions in the denominator.

\[
\frac{\cos x}{1 - \sin x}
\]

Multiply numerator and denominator by the conjugate of the denominator.

\[
\frac{\cos x}{1 - \sin x} \cdot \frac{1 - \sin x}{1 - \sin x}
\]

Multiply.

\[
\frac{\cos x}{1 - \sin^2 x}
\]

Pythagorean Identity

\[
\frac{\cos x}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x}
\]

Write as a sum of two fractions.

\[
\frac{1}{\cos x} + \frac{\sin x}{\cos x}
\]

Divide out the common factor \( \cos x \).

\[
\sec x + \tan x \checkmark
\]

**Reciprocal and Quotient Identities.**
5-2 Verifying Trigonometric Identities

36. \( \sec^2 x - 2 \sec x \tan x + \tan^2 x = \frac{1 - \cos x}{1 + \cos x} \)

**SOLUTION:**

Graph \( Y_1 = \sec^2 x - 2 \sec x \tan x + \tan^2 x \) and then graph \( Y_2 = \frac{1 - \cos x}{1 + \cos x} \).

The graphs of the related functions do not coincide for all values of \( x \) for which both functions are defined. Using the CALC feature on the graphing calculator to find that when \( x = 0 \), \( Y_1 = 1 \) and \( Y_2 = 0 \). Therefore, the equation is not an identity.

37. \( \frac{\cot 2x}{1 + \cot 2x} = 1 - 2 \sin^2 x \)

**SOLUTION:**

Graph \( Y_1 = \frac{\cot 2x}{1 + \cot 2x} \) and then graph \( Y_2 = 1 - 2 \sin^2 x \).

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

\[
\frac{\cot^2 x - 1}{1 + \cot^2 x} = \frac{\cot^2 x - 1}{\csc^2 x} = \frac{\cos^2 x}{\sin^2 x} - 1 = \frac{1}{\sin^2 x} - 1 = \frac{\cos^2 x}{\sin^2 x} = \left(\frac{\cos^2 x}{\sin^2 x} - 1\right)\sin^2 x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2 \sin^2 x \checkmark
\]

Pythagorean Identity
Quotient and Reciprocal Identities
Multiply and divide out common factor.
Pythagorean Identity
Simplify.
5-2 Verifying Trigonometric Identities

38. \(\frac{\tan x \sec x}{\tan x + \sec x} = \frac{\tan 2x I}{\sec 2x}\)

**SOLUTION:**

Graph \(Y_1 = \frac{\tan x \sec x}{\tan x + \sec x}\) and then graph \(Y_2 = \frac{\tan 2x I}{\sec 2x}\).

The graphs of the related functions do not coincide for all values of \(x\) for which both functions are defined.

Using the CALC feature on the graphing calculator to find that when \(x = \frac{\pi}{4}\), \(Y_1 \approx -0.17\) and \(Y_2 = 0\). Therefore, the equation is not an identity.
39. \( \cos^2 x - \sin^2 x = \frac{\cot \tan x}{\tan x + \cot x} \)

**SOLUTION:**

Graph \( Y_1 = \cos^2 x - \sin^2 x \) and then graph \( Y_2 = \frac{\cot \tan x}{\tan x + \cot x} \).

The equation *appears* to be an identity because the graphs of the related functions coincide. Verify this algebraically.

\[
\frac{\cot x - \tan x}{\tan x + \cot x} = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} \quad \text{Quotient Identities}
\]

\[
= \frac{\cos^2 x}{\sin^2 x + \cos^2 x} - \frac{\sin^2 x}{\sin^2 x + \cos^2 x} \quad \text{Multiply to change to common denominators.}
\]

\[
= \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x} \quad \text{Add fractions in numerator and denominator.}
\]

\[
= \frac{\cos^2 x - \sin^2 x}{\sin^2 x + \cos^2 x} \quad \text{Multiply numerator and denominator by (sin \ x \cdot \cos \ x).}
\]

\[
= \cos^2 x - \sin^2 x \quad \text{Pythagorean Identity}
\]
5-2 Verifying Trigonometric Identities

Verify each identity.

40. \[ \sqrt{\frac{\sin x \tan x}{\sec x}} = \sin x \]

**SOLUTION:**

\[
\sqrt{\frac{\sin x \tan x}{\sec x}} = \sqrt{\frac{\sin x \left(\frac{\sin x}{\cos x}\right)}{\cos x}}
\]

Quotient Identities  
Multiply \((\sin x)(\sin x)\).

\[
= \sqrt{\frac{\sin^2 x}{\cos x}}
\]

Multiply by reciprocal of denominator.

\[
= \sqrt{\frac{\sin^2 x \cos x}{\cos x}}
\]

Divide out the common factor of \(\cos x\).

\[
= |\sin x| \checkmark
\]

Simplify the square root.

41. \[ \sqrt{\frac{\sec^2 x}{\tan x}} = \sec x \]

**SOLUTION:**

\[
\sqrt{\frac{\sec x - 1}{\sec x + 1}} = \sqrt{\frac{\sec x - 1}{\sec x + 1}} \cdot \sqrt{\frac{\sec x - 1}{\sec x - 1}}
\]

Multiply numerator and denominator by conjugate of the denominator.

\[
= \sqrt{\frac{(\sec x - 1)^2}{\sec^2 x - 1}}
\]

Multiply in the numerator and denominator.

\[
= \sqrt{\frac{(\sec x - 1)^2}{\tan^2 x}}
\]

Pythagorean Identity

\[
= \frac{\sec x - 1}{\tan x} \checkmark
\]

Simplify the square root of numerator and denominator.

42. \[ \ln \csc x + \cot x + \ln \csc x - \cot x = 0 \]

**SOLUTION:**

\[
\ln |\csc x + \cot x| + \ln |\csc x - \cot x|
\]

Product Property of Logarithms

\[
= \ln (|\csc x + \cot x|)(|\csc x - \cot x|)
\]

Multiply.

\[
= \ln |\csc^2 x - \cot^2 x|
\]

Pythagorean Identity

\[
= \ln |1| \checkmark
\]

Simplify.
5-2 Verifying Trigonometric Identities

43. \[ \ln \cot x + \ln \tan x \cos x = \ln \cos x \]

**SOLUTION:**

\[
\begin{align*}
\ln |\cot x| + \ln |\tan x \cos x| &= \ln \left| \frac{\cos x}{\sin x} \right| + \ln \left| \frac{\sin x}{\cos x} \right| \\
&= \ln \left| \frac{\cos x}{\sin x} \right| + \ln |\sin x| \\
&= \ln \left| \frac{\cos x}{\sin x} \cdot \sin x \right| \\
&= \ln |\cos x| \checkmark
\end{align*}
\]

**Verify each identity.**

44. \[ \sec^2 \theta + \tan^2 \theta = \sec^4 \theta - \tan^2 \theta \]

**SOLUTION:**

Start with the right side of the identity.

\[
\begin{align*}
\sec^4 \theta - \tan^2 \theta &= (\sec^2 \theta + \tan^2 \theta)(\sec^2 \theta - \tan^2 \theta) \\
&= (\sec^2 \theta + \tan^2 \theta)(\tan^2 \theta + 1 - \tan^2 \theta) \\
&= (\sec^2 \theta + \tan^2 \theta)(1) \\
&= \sec^2 \theta + \tan^2 \theta \checkmark
\end{align*}
\]

45. \[ -2 \cos^2 \theta = \sin^4 \theta - \cos^4 \theta - 1 \]

**SOLUTION:**

Start with the right side of the identity.

\[
\begin{align*}
\sin^4 \theta - \cos^4 \theta - 1 &= (\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) - 1 \\
&= (1)(1 - \cos^2 \theta - \cos^2 \theta) - 1 \\
&= 1 - \cos^2 \theta - \cos^2 \theta - 1 \\
&= -2 \cos^2 \theta \checkmark
\end{align*}
\]
5-2 Verifying Trigonometric Identities

46. \( \sec^2 \theta \sin^2 \theta = \sec^4 \theta - (\tan^4 \theta + \sec^2 \theta) \)

**SOLUTION:**
Start with the right side of the identity.

\[
\begin{align*}
\sec^4 \theta - (\tan^4 \theta + \sec^2 \theta) \\
= (\sec^4 \theta - \tan^4 \theta) - \sec^2 \theta \\
= (\sec^2 \theta - \tan^2 \theta)(\sec^2 \theta + \tan^2 \theta) - \sec^2 \theta \\
= (\sec^2 \theta + \tan^2 \theta) - \sec^2 \theta \\
= \sec^2 \theta + \tan^2 \theta - \sec^2 \theta \\
= \tan^2 \theta \\
= \frac{\sin^2 \theta}{\cos^2 \theta} \\
= \frac{1 \cdot \sin^2 \theta}{\cos^2 \theta} \\
= \sec^2 \theta \sin^2 \theta \checkmark
\end{align*}
\]

Distributive and Associative Properties of Addition
Factor.
Pythagorean Identity
Multiply.
Add.
Quotient Identity
Write as product of two fractions.
Reciprocal Identity
5-2 Verifying Trigonometric Identities

47. \( 3 \sec^2 \theta \tan^2 \theta + 1 = \sec^6 \theta - \tan^6 \theta \)

**SOLUTION:**
Start with the right side of the identity.
\[
\sec^6 \theta - \tan^6 \theta = (\sec^3 \theta - \tan^3 \theta)(\sec^3 \theta + \tan^3 \theta)
\]
Factor difference of squares.
\[
= \sec^3 \theta (\sec \theta + \tan \theta)(\sec^2 \theta - \sec \theta \tan \theta + \tan^2 \theta)
\]
Factor sum and difference of cubes.
\[
= \sec \theta - \tan \theta)(\sec \theta + \tan \theta)(\sec^2 \theta + \sec \theta \tan \theta + \tan^2 \theta)
\]
Commutative Property of Multiplication
\[
= (1)(1 + 2 \tan^2 \theta + \sec \theta \tan \theta)
\]
Pythagorean Identity and Addition
\[
= (1 + 2 \tan^2 \theta)^2 - (\sec \theta \tan \theta)^2
\]
Product of sum and difference of two terms.
\[
= 1 + 4 \tan^2 \theta + 4 \tan^4 \theta - \sec^2 \theta \tan^2 \theta
\]
Square each expression.
\[
= 1 + (4 \tan^2 \theta + 4 \tan^4 \theta - \sec^2 \theta \tan^2 \theta)
\]
Associative Property of Addition
\[
= 1 + \tan^2 \theta(4 + 4 \sec^2 \theta - \sec^2 \theta)
\]
Factor.
\[
= 1 + \tan^2 \theta(4 + 4 \sec^2 \theta - \sec^2 \theta)
\]
Pythagorean Identity
\[
= 1 + \tan^2 \theta(4 + 4 \sec^2 \theta - 4 - \sec^2 \theta)
\]
Distributive Property
\[
= 1 + \tan^2 \theta(3 \sec^2 \theta)
\]
Combine like terms.
\[
= 1 + 3 \tan^2 \theta \sec^2 \theta
\]
Multiply.
\[
= 3 \sec^2 \theta \tan^2 \theta + 1
\]
Commutative Property of Addition

48. \( \sec^4 x = 1 + 2 \tan^2 x + \tan^4 x \)

**SOLUTION:**
Start with the right side of the identity.
\[
1 + 2 \tan^2 x + \tan^4 x
\]
\[
= 1 + 2(\sec^2 x - 1) + (\sec^2 x - 1)^2
\]
Pythagorean Identities
\[
= 1 + 2 \sec^2 x - 2 + \sec^4 x - 2 \sec^2 x + 1
\]
Distribute and square.
\[
= \sec^4 x
\]
Combine like terms.
5-2 Verifying Trigonometric Identities

49. \( \sec^2 x \csc^2 x = \sec^2 x + \csc^2 x \)

**SOLUTION:**
Start with the left side of the identity.

\[
\sec^2 x \csc^2 x
\]

\[
= (\tan^2 x + 1)\csc^2 x \quad \text{Pythagorean Identity}
\]

\[
= \left( \frac{\sin^2 x}{\cos^2 x} + 1 \right) \left( \frac{1}{\sin^2 x} \right) \quad \text{Quotient and Reciprocal Identities}
\]

\[
= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}
\]

Multiply.

\[
= \sec^2 x + \csc^2 x \quad \checkmark \quad \text{Reciprocal Identities}
\]

50. **Environment** A biologist studying pollution situates a net across a river and positions instruments at two different stations on the river bank to collect samples. In the diagram shown, \( d \) is the distance between the stations and \( w \) is width of the river.

![Diagram of river and stations](image)

a. Determine an equation in terms of tangent \( \alpha \) that can be used to find the distance between the stations.

b. Verify that \( d = \frac{w \cos(90^\circ - \alpha)}{\cos \alpha} \).

c. Complete the table shown for \( d = 40 \) feet.

<table>
<thead>
<tr>
<th>( w )</th>
<th>( 20 )</th>
<th>( 40 )</th>
<th>( 60 )</th>
<th>( 80 )</th>
<th>( 100 )</th>
<th>( 120 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
<td>( \alpha )</td>
</tr>
</tbody>
</table>

\[ d = \frac{w \cos(90^\circ - \alpha)}{\cos \alpha} \]

\[ d = w \tan \alpha \]

\[ d' = w \tan \alpha \]

\[ d' = \frac{w \sin \alpha}{\cos \alpha} \quad \text{Quotient Identity} \]

\[ = \frac{w \cos(90^\circ - \alpha)}{\cos \alpha} \quad \text{Cofunction Identity} \]

d. If \( \alpha > 60^\circ \) or \( \alpha < 20^\circ \), the instruments will not function properly. Use the table from part c to determine whether sites in which the width of the river is 5, 35, or 140 feet could be used for the experiment.

**SOLUTION:**

a. 

\[ \tan \alpha = \frac{\text{opp}}{\text{adj}} \quad \text{Tangent ratio} \]

\[ \tan \alpha = \frac{d}{w} \quad \text{opp} = d \text{ and adj} = w \]

\[ w \tan \alpha = d \quad \text{Multiply each side by} \ w. \]

\[ d = w \tan \alpha \quad \text{Symmetric Property of Equality} \]

b. 

\[ d' = w \tan \alpha \]

\[ = \frac{w \sin \alpha}{\cos \alpha} \quad \text{Quotient Identity} \]

\[ = \frac{w \cos(90^\circ - \alpha)}{\cos \alpha} \quad \text{Cofunction Identity} \]

c.
5-2 Verifying Trigonometric Identities

\[
\tan \alpha = \frac{d}{w}
\]
\[
\alpha = \tan^{-1} \left( \frac{d}{w} \right)
\]

\[
\begin{align*}
\alpha &= \tan^{-1} \left( \frac{40}{20} \right) \approx 63.4 & \alpha &= \tan^{-1} \left( \frac{40}{80} \right) \approx 26.6 \\
\alpha &= \tan^{-1} \left( \frac{40}{40} \right) = 45 & \alpha &= \tan^{-1} \left( \frac{40}{100} \right) \approx 21.8 \\
\alpha &= \tan^{-1} \left( \frac{40}{60} \right) \approx 33.7 & \alpha &= \tan^{-1} \left( \frac{40}{120} \right) \approx 18.4
\end{align*}
\]

<table>
<thead>
<tr>
<th>(w)</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha)</td>
<td>63.4</td>
<td>45</td>
<td>33.7</td>
<td>26.6</td>
<td>21.8</td>
<td>18.4</td>
</tr>
</tbody>
</table>

\text{d. Sample answer: If } w = 5 \text{ then } \alpha \text{ will be greater than } 63.4^\circ \text{ since } 5 < 20. \text{ If } w = 140, \text{ then } \alpha \text{ will be less than } 18.4^\circ \text{ since } 140 > 120. \text{ If } w = 35, \text{ then } 45^\circ < \alpha < 63.4^\circ \text{ since } 35 \text{ is between } 20 \text{ and } 40. \text{ The sites with widths of } 5 \text{ and } 140 \text{ feet could not be used because } \alpha > 60^\circ \text{ and } \alpha < 20^\circ, \text{ respectively. The site with a width of } 35 \text{ feet could be used because } 20^\circ < \alpha < 60^\circ. \]
5-2 Verifying Trigonometric Identities

HYPERBOLIC FUNCTIONS The hyperbolic trigonometric functions are defined in the following ways.

\[
\sinh x = \frac{1}{2} (e^x - e^{-x}) \\
\cosh x = \frac{1}{2} (e^x + e^{-x}) \\
\tanh x = \frac{\sinh x}{\cosh x} \\
\cosh x = \frac{1}{2} (e^x + e^{-x}) \\
\csch x = \frac{1}{\sinh x}, \quad x \neq 0 \\
\text{coth} x = \frac{1}{\tanh x}, \quad x \neq 0
\]

Verify each identity using the functions shown above.

51. \( \cosh^2 x - \sinh^2 x = 1 \)

**SOLUTION:**

\[
\begin{align*}
\cosh^2 x - \sinh^2 x & \quad \text{Start with the left side.} \\
= \frac{1}{4} (e^x + e^{-x})^2 - \frac{1}{4} (e^x - e^{-x})^2 & \quad \text{Replace \( \cosh \) and \( \sinh \) with definitions.} \\
= \frac{1}{4} [e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})] & \quad \text{Factor and square each expression.} \\
= \frac{1}{4} [e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}] & \quad \text{Distribute the negative.} \\
= \frac{1}{4} (4) & \quad \text{Combine like terms.} \\
= 1 & \quad \text{Multiply.}
\end{align*}
\]

52. \( \sinh (-x) = -\sinh x \)

**SOLUTION:**

\[
\begin{align*}
\sinh(-x) & \quad \text{Start with the left side.} \\
= \frac{1}{2} [e^{-x} - e^{-(x)}] & \quad \text{Substitute \(-x\) for \(x\) in definition for \(\sinh\).} \\
= \frac{1}{2} (e^{-x} - e^x) & \quad \text{Simplify.} \\
= \frac{1}{2} (-e^x + e^{-x}) & \quad \text{Commutative Property of Addition} \\
= -\frac{1}{2} (e^x - e^{-x}) & \quad \text{Factor out \(-1\).} \\
= -\left[ \frac{1}{2} (e^x - e^{-x}) \right] & \quad \text{Associative Property of Multiplication} \\
= -\sinh x & \quad \text{Substitute.}
\end{align*}
\]
5-2 Verifying Trigonometric Identities

53. \( \text{sech}^2 x = 1 - \text{tanh}^2 x \)

\textbf{SOLUTION:}

\[
1 - \text{tanh}^2 x = 1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x \checkmark
\]

Start with the right side.
Replace \( \tanh \) with \( \frac{\sinh}{\cosh} \).
Change to common denominators.
Add fractions.
From Problem 51, \( \cosh^2 x - \sinh^2 x = 1 \).
Replace \( \frac{1}{\cosh x} \) with \( \text{sech} x \).

54. \( \cosh (-x) = \cosh x \)

\textbf{SOLUTION:}

\[
\cosh (-x) = \frac{1}{2} \left[ e^{-x} + e^{(-(-x))} \right] = \frac{1}{2} (e^{-x} + e^x) = \frac{1}{2} (e^x + e^{-x}) = \cosh x \checkmark
\]

Start with the left side.
Substitute \( -x \) for \( x \) in definition for \( \cosh \).
Simplify.
Commutative Property of Addition
Substitute.
5-2 Verifying Trigonometric Identities

GRAPHING CALCULATOR Graph each side of each equation. If the equation appears to be an identity, verify it algebraically.

55. \[
\frac{\sec x}{\cos x} - \frac{\tan x \sec x}{\csc x} = 1
\]

**SOLUTION:**

Graph \( Y1 = \frac{\sec x}{\cos x} - \frac{\tan x \sec x}{\csc x} \) and \( Y2 = 1 \).

The graphs appear to be the same, so the equation appears to be an identity. Verify this algebraically.

\[
\begin{align*}
\frac{\sec x}{\cos x} - \frac{\tan x \sec x}{\csc x} &= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} - \frac{1}{\csc x} \cdot \frac{\cos x}{\cos x} \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos^2 x} - \frac{1}{\sin x} \cdot \frac{\cos x}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} - \frac{\cos x}{\cos^2 x} \\
&= \sec^2 x - \tan^2 x \\
&= 1
\end{align*}
\]

Start with the left side of the identity.

Quotient and Reciprocal Identities

Multiply fractions.

Multiply by reciprocal of the denominator.

Reciprocal and Quotient Identity

Pythagorean Identity
5-2 Verifying Trigonometric Identities

56. \[ \sec x - \cos^2 x \csc x = \tan x \sec x \]

\textit{SOLUTION:}

\[
\begin{align*}
[\alpha, 2\pi] \text{ scl: } & \frac{\pi}{2} \text{ by } [4, 4] \text{ scl: 1} \quad [\alpha, 2\pi] \text{ scl: } & \frac{\pi}{2} \text{ by } [-4, 4] \text{ scl: 1} \\
y = \sec x - \cos^2 x \csc x & \text { Graphs are not the same so } \sec x - \\
\cos^2 x \csc x \neq \tan x \sec x. \\
\end{align*}
\]

57. \[(\tan x + \sec x)(1 - \sin x) = \cos x \]

\textit{SOLUTION:}

\[
\begin{align*}
\tan x + \sec x)(1 - \sin x) & \text{ Start with the left side.} \\
= \tan x - \tan x \sin x + \sec x - \sec x \sin x & \text{ Multiply binomials.} \\
= \tan x - \frac{\sin x}{\cos x} \cdot \sin x + \frac{1}{\cos x} - \frac{1}{\cos x} \cdot \sin x & \text{ Quotient and Reciprocal Identities} \\
= \tan x - \frac{\sin^2 x}{\cos x} + \frac{1}{\cos x} - \frac{\sin x}{\cos x} & \text{ Multiply.} \\
= \tan x - \frac{\sin^2 x}{\cos x} + \frac{1}{\cos x} - \tan x & \text{ Quotient Identity} \\
= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x} & \text{ Commutative Property} \\
= \frac{1 - \sin^2 x}{\cos x} & \text{ Add fractions.} \\
= \frac{\cos^2 x}{\cos x} & \text{ Pythagorean Identity} \\
= \cos x & \text{ Divide out common factor.}
\end{align*}
\]
5-2 Verifying Trigonometric Identities

58. \[
\frac{\sec x \cos x}{\cot 2x} \frac{1}{\tan 2x \sin 2x \tan 2x} = -1
\]

**SOLUTION:**

\[
y = \frac{\sec x \cos x}{\cot^2 x} - \frac{1}{\tan^2 x - \sin^2 x \tan^2 x}
\]

\([-2\pi, 2\pi] \text{ scl: } \frac{\pi}{2} \text{ by } [-4, 4] \text{ scl: } 1\]

\([-2\pi, 2\pi] \text{ scl: } \frac{\pi}{2} \text{ by } [-4, 4] \text{ scl: } 1\]

The graphs are not the same, so \[
\frac{\sec x \cos x}{\cot^2 x} - \frac{1}{\tan^2 x - \sin^2 x \tan^2 x} \neq -1
\]

59. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate methods used to solve trigonometric equations. Consider \(1 = 2 \sin x\).

**a. NUMERICAL** Isolate the trigonometric function in the equation so that \(\sin x\) is the only expression on one side of the equation.

**b. GRAPHICAL** Graph the left and right sides of the equation you found in part a on the same graph over \([0, 2\pi]\). Locate any points of intersection and express the values in terms of radians.

**c. GEOMETRIC** Use the unit circle to verify the answers you found in part b.

**d. GRAPHICAL** Graph the left and right sides of the equation you found in part a on the same graph over \(-2\pi < x < 2\pi\). Locate any points of intersection and express the values in terms of radians.

**e. VERBAL** Make a conjecture as to the solutions of \(1 = 2 \sin x\). Explain your reasoning.

**SOLUTION:**

**a.** \(2 \sin x = 1\)

\[
\frac{2 \sin x}{2} = \frac{1}{2}
\]

\(\sin x = \frac{1}{2}\)

**b.**

The graphs of \(y = \sin x\) and \(y = \frac{1}{2}\) intersect at \(\frac{\pi}{6}\) and \(\frac{5\pi}{6}\) over \([0, 2\pi]\).
5-2 Verifying Trigonometric Identities

c.

\[
\begin{array}{c}
-\frac{\sqrt{3}}{2}, \frac{1}{2} \\
5\pi/6 \\
\pi/6 \\
\end{array}
\]

\[y = \sin x\text{, }y = \frac{1}{2}\]

The graphs of \(y = \sin x\) and \(y = \frac{1}{2}\) intersect at \(-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \text{ and } \frac{5\pi}{6}\) over \((-2\pi, 2\pi)\).

d.

\[y = \frac{1}{2}\]

\[y = \sin x\]

Over \([-2\pi, 2\pi]\) scaled by \[\frac{\pi}{2}\] and \([-2, 2]\) scaled by 1.

**e.** Sample answer: Since sine is a periodic function, the solutions of \(\sin x = \frac{1}{2}\) are

\[x = \frac{\pi}{6} + 2n\pi \text{ and } x = \frac{5\pi}{6} + 2n\pi, \text{ where } n \text{ is an integer.}\]

60. **REASONING** Can substitution be used to determine whether an equation is an identity? Explain your reasoning.

**SOLUTION:**

Sample answer: Substitution can be used to determine whether an equation is not an identity. However, this method cannot be used to determine whether an equation is an identity, because there is no way to prove that the identity is true for the entire domain.
5-2 Verifying Trigonometric Identities

61. **CHALLENGE** Verify that the area $A$ of a triangle is given by

$$A = \frac{a2\sin \beta \sin \gamma}{2 \sin (\beta + \gamma)},$$

where $a$, $b$, and $c$ represent the sides of the triangle and $\alpha$, $\beta$, and $\gamma$ are the respective opposite angles.

**SOLUTION:**

Using the Law of Sines, \( \frac{\sin \beta}{b} = \frac{\sin \alpha}{a} \), so $b = \frac{as\sin \beta}{\sin \alpha}$.

$$A = \frac{1}{2} ab \sin \gamma \quad \text{Area of a triangle given SAS}$$

$$A = \frac{1}{2} a \left( \frac{a \sin \beta}{\sin \alpha} \right) \sin \gamma \quad \text{Substitution}$$

$$A = \frac{a2\sin \beta \sin \gamma}{2 \sin \alpha} \quad \text{Multiply.}$$

$$A = \frac{a2\sin \beta \sin \gamma}{2 \sin[180^\circ(\beta + \gamma)]} \quad a + \beta + \gamma = 180^\circ, \text{ so } a = 180^\circ - (\beta + \gamma).$$

$$A = \quad \text{Sine Sum Identity}$$

$$A = \frac{a2\sin \beta \sin \gamma}{2[\cos(\beta + \gamma)(1)\sin(\beta + \gamma)]} \quad \sin 180^\circ = 0, \cos 180^\circ = -1$$

$$A = \frac{a2\sin \beta \sin \gamma}{2 \sin(\beta + \gamma)} \quad \text{Simplify.}$$

62. **Writing in Math** Use the properties of logarithms to explain why the sum of the natural logarithm of the six basic trigonometric functions for any angle $\theta$ is 0.

**SOLUTION:**

Sample answer: According to the Product Property of Logarithms, the sum of the logarithms of the basic trigonometric functions is equal to the logarithm of the product. Since the product of the absolute values of the functions is 1, the sum of the logarithms is $\ln 1$ or 0.
5-2 Verifying Trigonometric Identities

63. **OPEN ENDED** Create identities for sec \(x\) and csc \(x\) in terms of two or more of the other basic trigonometric functions.

**SOLUTION:**
Sample answers: \(\tan x \sin x + \cos x = \sec x\) and \(\sin x + \cot x \cos x = \csc x\)

\[
\tan x \sin x + \cos x = \frac{\sin x}{\cos x} \cdot \sin x + \cos x
\]
\[
= \frac{\sin^2 x}{\cos x} + \cos x
\]
\[
= \frac{1 - \cos^2 x}{\cos x} + \cos x
\]
\[
= \frac{1}{\cos x} - \cos x + \cos x
\]
\[
= \frac{1}{\cos x}
\]
\[
= \sec x
\]

\[
\sin x + \cot x \cos x = \sin x + \frac{\cos x}{\sin x} \cdot \cos x
\]
\[
= \sin x + \frac{\cos^2 x}{\sin x}
\]
\[
= \sin x + \frac{1 - \sin^2 x}{\sin x}
\]
\[
= \sin x + \frac{1}{\sin x} - \sin x
\]
\[
= \frac{1}{\sin x}
\]
\[
= \csc x
\]

64. **REASONING** If two angles \(\alpha\) and \(\beta\) are complementary, is \(\cos^2 \alpha + \cos^2 \beta = 1\)? Explain your reasoning. Justify your answers.

**SOLUTION:**
Yes; sample answer: If \(\alpha\) and \(\beta\) are complementary angles, then \(\alpha + \beta = 90^\circ\)

\[
\cos^2 \alpha + \cos^2 \beta
\]
\[
= \cos^2 \alpha + \cos^2 (90^\circ - \alpha)
\]
\[
= \cos^2 \alpha + \sin^2 \alpha = 1.
\]

65. **Writing in Math** Explain how you would verify a trigonometric identity in which both sides of the equation are equally complex.

**SOLUTION:**
Sample answer: You could start on the left side of the identity and simplify it as much as possible. Then, you could move to the right side and simplify until it matches the left side.
5-2 Verifying Trigonometric Identities

Simplify each expression.

66. **cos \( \theta \) csc \( \theta \)**

   **SOLUTION:**
   
   \[
   \cos \theta \csc \theta = \cos \theta \cdot \frac{1}{\sin \theta} \\
   = \frac{\cos \theta}{\sin \theta} \\
   = \cot \theta
   \]

   \( \text{Reciprocal Identity} \)
   
   \( \text{Multiply.} \)
   
   \( \text{Quotient Identity} \)

67. **tan \( \theta \) cot \( \theta \)**

   **SOLUTION:**
   
   \[
   \tan \theta \cot \theta = \frac{\sin \theta \cdot \cos \theta}{\cos \theta \cdot \sin \theta} \\
   = \frac{1}{1} \\
   = 1
   \]

   \( \text{Quotient Identities} \)
   
   \( \text{Divide out the common factors.} \)

68. **sin \( \theta \) cot \( \theta \)**

   **SOLUTION:**
   
   \[
   \sin \theta \cot \theta = \sin \theta \cdot \frac{\cos \theta}{\sin \theta} \\
   = \cos \theta
   \]

   \( \text{Quotient Identity} \)
   
   \( \text{Divide out the common factors.} \)

69. **\( \frac{\cos \theta \csc \theta}{\tan \theta} \)**

   **SOLUTION:**
   
   \[
   \frac{\cos \theta \csc \theta}{\tan \theta} = \frac{\cos \theta \cdot 1}{\frac{1}{\sin \theta}} \\
   = \frac{\cos \theta}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta} \\
   = \frac{\cos^2 \theta}{\sin^2 \theta} \\
   = \cot^2 \theta
   \]

   \( \text{Reciprocal and Quotient Identities} \)
   
   \( \text{Multiply by reciprocal of the denominator.} \)
   
   \( \text{Multiply fractions.} \)
   
   \( \text{Quotient Identity} \)
5-2 Verifying Trigonometric Identities

70. \( \frac{\sin \theta \csc \theta}{\cot \theta} \)

*SOLUTION:*

\[
\frac{\sin \theta \, \csc \theta}{\cot \theta} = \frac{\sin \theta \cdot 1}{\frac{\cos \theta}{\sin \theta}} = \frac{1}{\frac{\cos \theta}{\sin \theta}} \cdot \frac{\sin \theta}{1} = \frac{\sin \theta}{\cos \theta} = \tan \theta
\]

Reciprocal and Quotient Identities

71. \( \frac{1-\cos 2\theta}{\sin 2\theta} \)

*SOLUTION:*

\[
\frac{1-\cos^2 \theta}{\sin^2 \theta} = \frac{(\sin^2 \theta + \cos^2 \theta) - \cos^2 \theta}{\sin^2 \theta} = \frac{\sin^2 \theta}{\sin^2 \theta} = 1
\]

Pythagorean Identity

Combine like terms.

Divide out the common factor of \( \sin^2 \theta \).
5-2 Verifying Trigonometric Identities

72. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts, the angles of depression are 64° and 7°. How high is the balloon to the nearest foot?

\[ \begin{align*}
\text{SOLUTION:} \\
\text{First, find the measures of } \angle CAD, \angle BCA, \angle ACD \text{ and } \angle DAE. \\
m_\angle CAD &= 64° - 7° \\
&= 57° \\
m_\angle BCA &= 180° - (7° + 90°) \\
&= 83° \\
m_\angle ACD &= 90° - 83° \\
&= 7° \\
m_\angle DAE &= 90° - 64° \\
&= 26° \\
\text{In } \triangle ACD, \text{ use the law of sines to find the length of } \overline{AD}. \\
\sin 57° &= \frac{\sin 7°}{5280} \\
\overline{AD} &= \frac{5280 \sin 7°}{\sin 57°} \text{ or about 767.3 feet} \\
\text{Next, use right triangle } ADE \text{ and the cosine function to find the length of } \overline{AE}. \\
\cos 26° &= \frac{\overline{AE}}{\overline{AD}} \\
\cos 26° &= \frac{\overline{AE}}{767.3} \\
\overline{AE} &= 767.3 \cos 26° \text{ or about 690 ft}
\end{align*} \]
5-2 Verifying Trigonometric Identities

Locate the vertical asymptotes, and sketch the graph of each function.

73. \( y = \frac{1}{4} \tan x \)

\textbf{SOLUTION:}

The graph of \( y = \frac{1}{4} \tan x \) is the graph of \( y = \tan x \) compressed vertically. The period is \( \frac{\pi}{1} \) or \( \pi \). Find the location of two consecutive vertical asymptotes.

\begin{align*}
\frac{b}{x} + c &= -\frac{\pi}{2} & \frac{b}{x} + c &= \frac{\pi}{2} \\
(1)x + 0 &= -\frac{\pi}{2} & (1)x + 0 &= \frac{\pi}{2} \\
x &= -\frac{\pi}{2} & x &= \frac{\pi}{2}
\end{align*}

Create a table listing the coordinates of key points for \( y = \frac{1}{4} \tan x \) for one period on \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Vertical Asymptote</th>
<th>Intermediate Point</th>
<th>( x )-intercept</th>
<th>Intermediate Point</th>
<th>Vertical Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \tan x )</td>
<td>( x = -\frac{\pi}{2} )</td>
<td>( \left( -\frac{\pi}{4}, 1 \right) )</td>
<td>( (0, 0) )</td>
<td>( \left( \frac{\pi}{4}, 1 \right) )</td>
<td>( x = \frac{\pi}{2} )</td>
</tr>
<tr>
<td>( y = \frac{1}{4} \tan x )</td>
<td>( x = -\frac{\pi}{2} )</td>
<td>( \left( \frac{\pi}{4}, -\frac{1}{4} \right) )</td>
<td>( (0, 0) )</td>
<td>( \left( \frac{\pi}{4}, -\frac{1}{4} \right) )</td>
<td>( x = \frac{\pi}{2} )</td>
</tr>
</tbody>
</table>

Sketch the curve through the indicated key points for the function. Then repeat the pattern.
5-2 Verifying Trigonometric Identities

74. \( y = \csc 2x \)

**SOLUTION:**

The graph of \( y = \csc 2x \) is the graph of \( y = \csc x \) compressed horizontally. The period is \( \frac{2\pi}{2} \) or \( \pi \). Find the location of two vertical asymptotes.

\[
\begin{align*}
bx + c &= \frac{\pi}{2} & bx + c &= \frac{\pi}{2} \\
2x + 0 &= -\pi & 2x + 0 &= \pi
\end{align*}
\]

\[
\begin{align*}
x &= -\frac{\pi}{2} & x &= \frac{\pi}{2}
\end{align*}
\]

Create a table listing the coordinates of key points for \( y = \csc 2x \) for one period on \( \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Vertical Asymptote</th>
<th>Intermediate Point</th>
<th>Vertical Asymptote</th>
<th>Intermediate Point</th>
<th>Vertical Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \csc x )</td>
<td>( x = -\pi )</td>
<td>( \left(-\frac{\pi}{2}, -1\right) )</td>
<td>( x = 0 )</td>
<td>( \left(\frac{\pi}{2}, 1\right) )</td>
<td>( x = \pi )</td>
</tr>
<tr>
<td>( y = \csc 2x )</td>
<td>( x = \frac{\pi}{2} )</td>
<td>( \left(-\frac{\pi}{4}, -1\right) )</td>
<td>( x = 0 )</td>
<td>( \left(\frac{\pi}{4}, 1\right) )</td>
<td>( x = \frac{\pi}{2} )</td>
</tr>
</tbody>
</table>

Sketch the curve through the indicated key points for the function. Then repeat the pattern.
5-2 Verifying Trigonometric Identities

75. \( y = \frac{1}{2} \sec 3x \)

**SOLUTION:**

The graph of \( y = \frac{1}{2} \sec 3x \) is the graph of \( y = \sec x \) compressed vertically and horizontally. The period is \( \frac{2\pi}{3} \), so find the location of two vertical asymptotes.

\[
\begin{align*}
3x + 0 &= \frac{\pi}{2} \\
3x &= \frac{\pi}{2} \\
x &= \frac{\pi}{6}
\end{align*}
\]

\[
\begin{align*}
3x + 0 &= \frac{3\pi}{2} \\
3x &= \frac{3\pi}{2} \\
x &= \frac{\pi}{2}
\end{align*}
\]

Create a table listing the coordinates of key points for \( y = \frac{1}{2} \sec 3x \) for one period on \( \left[-\frac{\pi}{6}, \frac{\pi}{2}\right] \).

<table>
<thead>
<tr>
<th>Function</th>
<th>Vertical Asymptote</th>
<th>Intermediate Point</th>
<th>Vertical Asymptote</th>
<th>Intermediate Point</th>
<th>Vertical Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \sec x )</td>
<td>( x = -\frac{\pi}{2} )</td>
<td>(0,1)</td>
<td>( x = \frac{\pi}{2} )</td>
<td>(( \pi ), -1)</td>
<td>( x = \frac{3\pi}{2} )</td>
</tr>
<tr>
<td>( y = \frac{1}{2} \sec 3x )</td>
<td>( x = -\frac{\pi}{6} )</td>
<td>( \left(0, \frac{1}{2}\right) )</td>
<td>( x = \frac{\pi}{3} )</td>
<td>( \left(\frac{\pi}{3}, -\frac{1}{2}\right) )</td>
<td>( x = \frac{\pi}{2} )</td>
</tr>
</tbody>
</table>

Sketch the curve through the indicated key points for the function. Then repeat the pattern.

Write each degree measure in radians as a multiple of \( \pi \) and each radian measure in degrees.

76. \( 660^\circ \)

**SOLUTION:**

\[
660^\circ = 660 \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{11\pi}{3} \text{ radians or } 380^\circ \quad \text{Simplify.}
\]
5-2 Verifying Trigonometric Identities

77. 570°

\[ \text{SOLUTION:} \]
\[ 570° = 570 \left( \frac{\pi \text{ radians}}{180°} \right) \]
\[ = \frac{19\pi}{6} \text{ radians or } \frac{19\pi}{6} \text{ radians} \]
\[ \text{Multiply by } \frac{\pi \text{ radians}}{180°}. \]
\[ \text{Simplify.} \]

78. 158°

\[ \text{SOLUTION:} \]
\[ 158° = 158 \left( \frac{\pi \text{ radians}}{180°} \right) \]
\[ = \frac{79\pi}{90} \text{ radians or } \frac{79\pi}{90} \text{ radians} \]
\[ \text{Multiply by } \frac{\pi \text{ radians}}{180°}. \]
\[ \text{Simplify.} \]

79. \( \frac{29\pi}{4} \)

\[ \text{SOLUTION:} \]
\[ \frac{29\pi}{4} = \frac{29\pi}{4} \text{ radians} \]
\[ = \frac{29\pi}{4} \text{ radians} \left( \frac{180°}{\pi \text{ radians}} \right) \]
\[ = \frac{1305°}{4} \text{ radians} \]
\[ \text{Multiply by } \frac{180°}{\pi \text{ radians}}. \]
\[ \text{Simplify.} \]

80. \( \frac{17\pi}{6} \)

\[ \text{SOLUTION:} \]
\[ \frac{17\pi}{6} = \frac{17\pi}{6} \text{ radians} \]
\[ = \frac{17\pi}{6} \text{ radians} \left( \frac{180°}{\pi \text{ radians}} \right) \]
\[ = 510° \text{ radians} \]
\[ \text{Multiply by } \frac{180°}{\pi \text{ radians}}. \]
\[ \text{Simplify.} \]

81. 9

\[ \text{SOLUTION:} \]
\[ 9 = 9 \text{ radians} \]
\[ = 9 \text{ radians} \left( \frac{180°}{\pi \text{ radians}} \right) \]
\[ = \left( \frac{1620°}{\pi} \right) \text{ radians} \]
\[ \approx 515.7° \]
\[ \text{Multiply by } \frac{180°}{\pi \text{ radians}}. \]
\[ \text{Simplify.} \]
\[ \text{Divide.} \]
5-2 Verifying Trigonometric Identities

Solve each inequality.

82. \( x^2 - 3x - 18 > 0 \)

**SOLUTION:**
Let \( f(x) = x^2 - 3x - 18 \)
\[ = (x + 3)(x - 6) \]
f(x) has real zeros at \( x = -3 \) and \( x = 6 \). Set up a sign chart. Substitute an x-value in each test interval into the polynomial to determine if \( f(x) \) is positive or negative at that point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(6)</th>
<th>() +</th>
<th>(-)</th>
<th>() +</th>
</tr>
</thead>
</table>

The solutions of \( x^2 - 3x - 18 > 0 \) are \( x \)-values such that \( f(x) \) is positive. From the sign chart, you can see that the solution set is \(( -\infty, -3 \) \( ) \cup (6, \infty) \).

83. \( x^2 + 3x - 28 < 0 \)

**SOLUTION:**
Let \( f(x) = x^2 + 3x - 28 \)
\[ = (x + 7)(x - 4) \]
f(x) has real zeros at \( x = -7 \) and \( x = 4 \). Set up a sign chart. Substitute an \( x \)-value in each test interval into the polynomial to determine if \( f(x) \) is positive or negative at that point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-7)</th>
<th>(4)</th>
<th>() +</th>
<th>(-)</th>
<th>() +</th>
</tr>
</thead>
</table>

The solutions of \( x^2 + 3x - 28 < 0 \) are \( x \)-values such that \( f(x) \) is negative. From the sign chart, you can see that the solution set is \((-7, 4) \).

84. \( x^2 - 4x \leq 5 \)

**SOLUTION:**
First, write \( x^2 - 4x \leq 5 \) as \( x^2 - 4x - 5 \leq 0 \).
Let \( f(x) = x^2 - 4x - 5 \)
\[ = (x + 1)(x - 5) \]
f(x) has real zeros at \( x = -1 \) and \( x = 5 \). Set up a sign chart. Substitute an \( x \)-value in each test interval into the polynomial to determine if \( f(x) \) is positive or negative at that point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>(5)</th>
<th>() +</th>
<th>(-)</th>
<th>() +</th>
</tr>
</thead>
</table>

The solutions of \( x^2 - 4x - 5 \leq 0 \) are \( x \)-values such that \( f(x) \) is negative or equal to 0. From the sign chart, you can see that the solution set is \([-1, 5]\).
5-2 Verifying Trigonometric Identities

85. \( x^2 + 2x \geq 24 \)

\textit{SOLUTION:}

First, write \( x^2 + 2x \geq 24 \) as \( x^2 + 2x - 24 \geq 0 \).

Let \( f(x) = x^2 + 2x - 24 \)

\[
x^2 + 2x - 24 = (x + 6)(x - 4)
\]

\( f(x) \) has real zeros at \( x = -6 \) and \( x = 4 \). Set up a sign chart. Substitute an \( x \)-value in each test interval into the polynomial to determine if \( f(x) \) is positive or negative at that point.

\[
\begin{array}{c|c|c|c}
-6 & 4 & x \\
(+) & (-) & (+)
\end{array}
\]

The solutions of \( x^2 + 2x - 24 \geq 0 \) are \( x \)-values such that \( f(x) \) is positive or equal to 0. From the sign chart, you can see that the solution set is \( (-\infty, -6] \cup [4, \infty) \).

86. \(-x^2 - x + 12 \geq 0\)

\textit{SOLUTION:}

First, write \(-x^2 - x + 12 \geq 0\) as \( x^2 + x - 12 \leq 0 \).

Let \( f(x) = x^2 + x - 12 \)

\[
x^2 + x - 12 = (x + 4)(x - 3)
\]

\( f(x) \) has real zeros at \( x = -4 \) and \( x = 3 \). Set up a sign chart. Substitute an \( x \)-value in each test interval into the polynomial to determine if \( f(x) \) is positive or negative at that point.

\[
\begin{array}{c|c|c|c}
-4 & 3 & x \\
(+) & (-) & (+)
\end{array}
\]

The solutions of \( x^2 + x - 12 \leq 0 \) are \( x \)-values such that \( f(x) \) is negative or equal to 0. From the sign chart, you can see that the solution set is \([ -4, 3 ]\).
87. \(-x^2 - 6x + 7 \leq 0\)

**SOLUTION:**
First, write \(-x^2 - 6x + 7 \leq 0\) as \(x^2 + 6x - 7 \geq 0\).
Let \(f(x) = x^2 + 6x - 7\)
\[= (x + 7)(x - 1)\]
f(x) has real zeros at \(x = -7\) and \(x = 1\). Set up a sign chart. Substitute an \(x\)-value in each test interval into the polynomial to determine if \(f(x)\) is positive or negative at that point.

\[
\begin{array}{c|c|c|c}
& (-) & (+) & (+) \\
-7 & 1 & x \\
\end{array}
\]

The solutions of \(x^2 + 6x - 7 \geq 0\) are \(x\)-values such that \(f(x)\) is positive or equal to 0. From the sign chart, you can see that the solution set is \((-\infty, -7] \cup [1, \infty)\).

88. **FOOD** The manager of a bakery is randomly checking slices of cake prepared by employees to ensure that the correct amount of flavor is in each slice. Each 12-ounce slice should contain half chocolate and half vanilla flavored cream. The amount of chocolate by which each slice varies can be represented by \(g(x) = \frac{1}{2} |x - 12|\). Describe the transformations in the function. Then graph the function.

**SOLUTION:**
The parent function of \(g(x)\) is \(f(x) = |x|\). The factor of \(\frac{1}{2}\) will cause the graph to be compressed since \(\frac{1}{2} < 1\) and the subtraction of 12 will translate the graph 12 units to the right.
Make a table of values for \(x\) and \(g(x)\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>(g(x))</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Plot the points and draw the graph of \(g(x)\).
5-2 Verifying Trigonometric Identities

89. SAT/ACT

\[
a, b, a, b, a, b, b, a, b, b, a, b, b, a, \ldots
\]

If the sequence continues in this manner, how many \( b \)s are there between the 44th and 47th appearances of the letter \( a \)?

A 91
B 135
C 138
D 182
E 230

**SOLUTION:**
The number of \( b \)s after each \( a \) is the same as the number of \( a \) in the list (i.e., after the 44th \( a \) there are 44 \( b \)s).
Between the 44th and 47th appearances of \( a \) the number of \( b \)s will be 44 + 45 + 46 or 135.
Therefore, the correct answer choice is B.

90. Which expression can be used to form an identity with \( \frac{\sec \theta + \csc \theta}{1 + \tan \theta} \), when \( \tan \theta \neq -1 \)?

F \( \sin \theta \)
G \( \cos \theta \)
H \( \tan \theta \)
J \( \csc \theta \)

**SOLUTION:**

\[
\frac{\sec \theta + \csc \theta}{1 + \tan \theta} = \frac{1 + \frac{1}{\sin \theta}}{1 + \frac{1}{\cos \theta}}
\]

- **Reciprocal and Quotient Identities**

\[
= \frac{\sin \theta + \cos \theta}{\cos \theta + \sin \theta}
\]

- **Change fractions to common denominators.**

\[
= \cos \theta \sin \theta
\]

- **Add fractions.**

\[
= \cos \theta \sin \theta
\]

- **Multiply by reciprocal of the denominator.**

\[
= \frac{1}{\sin \theta}
\]

- **Divide out the common factors.**

Therefore, the correct answer choice is J.
5-2 Verifying Trigonometric Identities

91. **REVIEW** Which of the following is not equivalent to \( \cos \theta \), when \( 0 < \theta < \frac{\pi}{2} \)?

   A \[ \frac{\cos \theta}{\cos 2\theta + \sin 2\theta} \]
   B \[ \frac{1 - \sin 2\theta}{\cos \theta} \]
   C \[ \cot \theta \sin \theta \]
   D \[ \tan \theta \csc \theta \]

**SOLUTION:**

A. \[ \frac{\cos \theta}{\cos^2 \theta + \sin^2 \theta} = \frac{\cos \theta}{1} = \cos \theta \]

B. \[ \frac{1 - \sin^2 \theta}{\cos \theta} = \frac{\cos \theta}{\cos \theta} = 1 \]

C. \[ \cot \theta \sin \theta = \frac{\cos \theta}{\sin \theta} \cdot \sin \theta = \cos \theta \]

D. \[ \tan \theta \csc \theta = \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta} = \frac{1}{\cos \theta} \neq \cos \theta \]

Therefore, the correct answer choice is D.

92. **REVIEW** Which of the following is equivalent to \( \sin \theta + \cot \theta \cos \theta \)?

   F \[ 2 \sin \theta \]
   G \[ \frac{1}{\sin \theta} \]
   H \[ \cos^2 \theta \]
   J \[ \frac{\sin \theta + \cos \theta}{\sin 2\theta} \]

**SOLUTION:**

\[
\sin \theta + \cot \theta \cos \theta = \sin \theta + \frac{\cos \theta}{\sin \theta} \cdot \frac{\cos \theta}{1} \quad \text{Reciprocal Identity}
\]

\[= \sin \theta + \frac{\cos^2 \theta}{\sin \theta} \]

\[= \frac{\sin \theta^2 + \cos^2 \theta}{\sin \theta} \quad \text{Multiply.}
\]

\[= \frac{\sin \theta^2 + \cos^2 \theta}{\sin \theta} \quad \text{Multiply \( \sin \theta \) by \( \frac{\sin \theta}{\sin \theta} \).}
\]

\[= \frac{\sin \theta^2 + \cos^2 \theta}{\sin \theta} \quad \text{Add fractions.}
\]

\[= \frac{1}{\sin \theta} \quad \text{Pythagorean Identity}
\]

Therefore, the correct answer choice is G.