Chapter 8.1.1
Core Connections Algebra
Introduction to Factoring

Chapter 3 you learned how to multiply algebraic expressions using generic rectangles. This section will focus on reversing this process: How can you find a product when given a sum?

8-1. Review what you know about products and sums below.

a. Write the area of the rectangle at right as a product and as a sum. Remember that the product represents the area found by multiplying the length by the width, while the sum is the result of adding the areas inside the rectangle.

Product: \((x+y)(y+x+2)\)
Sum: \(xy + x^2 + 6x + 4y + 8\)

b. Use a generic rectangle to multiply \((6x - 1)(3x + 2)\). Write your solution as a sum.

\[
\begin{array}{cc|c|c|c}
2 & 12x & -2 & \\
3x & 18x^2 & -3x & \\
6x & -1 & \\
\end{array}
\]

Sum: \(18x^2 + 9x - 2\)

More Vocabulary for Expressions

Since algebraic expressions come in several different forms, there are special words used to help describe these expressions. For example, if the expression can be written in the form \(ax^2 + bx + c\) and \(a\) is not 0, then it is called a quadratic expression. Review the examples of quadratic expressions below.

\[x^2 - 15x + 26\]
\[16m^2 - 25\]
\[12 - 3k^2 + 5k\]

The way an expression is written can also be named. When an expression is written in product form, it is said to be factored. When factored, each of the expressions being multiplied is called a factor. For example, the factored form of \(x^2 - 15x + 26\) is \((x - 13)(x - 2)\), so \(x - 13\) and \(x - 2\) are each factors of the original expression.

Finally, if the expression is a polynomial (see Math Notes box in Lesson 3.2.3) the number of terms can help you name the polynomial. If the polynomial has one term, it is called a monomial, while a polynomial with two terms is called a binomial. If the polynomial has three terms, it is called a trinomial. Review the examples below.

Examples of monomials: \(15y^2\) and \(-2\)

Examples of binomials: \(16m - 25\) and \(7h^2 + \frac{1}{2}\)

Examples of trinomials: \(12 - 3k^2 + 5k\) and \(x^2 - 15x + 26\)
8-2. The process of changing a sum to a product is called **factoring**. Can every expression be factored? That is, *does every sum have a product that can be represented with tiles?*

Investigate this question by building rectangles with **algebra tiles** for the following expressions. For each one, write the area as a sum and as a product. If you cannot build a rectangle, be prepared to convince the class that no rectangle exists (and thus the expression cannot be factored).

a. \(2x^2 + 7x + 6\)
   
   \[\text{Product: } (2x + 3)(x + 2)\]

b. \(6x^2 + 7x + 2\)
   
   \[\text{Product: } (3x + 1)(2x + 2)\]

c. \(x^2 + 4x + 1\)
   
   **Not Possible**

d. \(2xy + 6x + y^2 + 3y\)
   
   \[\text{Product: } (x + y)(y + 3)\]

8-3. Work with your team to find the sum and the product for the following generic rectangles. Are there any special strategies you discovered that can help you determine the dimensions of the rectangle? Be sure to share these strategies with your teammates.

a. \[
\begin{array}{c|c|c}
3x & 2x & 5 \\
\hline
6x^2 & 15x \\
\hline
2x & 5
\end{array}
\]
   
   \[\text{Product: } (3x + 1)(2x + 5)\]
   
   \[\text{Sum: } 6x^2 + 17x + 5\]

b. \[
\begin{array}{c|c|c|c}
-3 & -2y & -6 \\
\hline
-9x & -12 \\
\hline
12x^2 & 16x \\
\hline
3x & 4
\end{array}
\]
   
   \[\text{Product: } (3x + y)(4x - 3)\]
   
   \[\text{Sum: } 12x^2 + 7x - 12\]

8-4. While working on problem 8-3, Casey noticed a pattern with the diagonals of each generic rectangle. However, just before she shared her pattern with the rest of her team, she was called out of class! The drawing on her paper looked like the diagram below. Can you figure out what the two diagonals have in common? Does Casey's pattern always work?

Verify that her pattern works for all of the generic rectangles in problem 8-3.

**The Product of each diagonal is equal.**

\[(2x)(15x) = 30x^2\]

\[(6x^2)(5) = 30x^2\]
Since mathematics is often described as the study of patterns, it is not surprising that generic rectangles have many patterns. You saw one important pattern in Lesson 8.1.1 (Casey's pattern from problem 8-4). Today you will continue to use patterns while you develop a method to factor trinomial expressions.

8-13. Examine the generic rectangle shown at the right.
   a. Review what you learned in Lesson 8.1.1 by writing the area of the rectangle at right as a sum and as a product.
      \[ \text{Product: } (2x-7)(5x-3) \]
      \[ \text{Sum: } 10x^2 - 39x + 14 \]

   b. Does this generic rectangle fit Casey's pattern for diagonals? Demonstrate that the product of each diagonal is equal.
      \[ (10x^2)(14) = 140x^2 \]
      \[ (-35x)(-4x) = 140x^2 \]

8-14. To develop a method for factoring without algebra tiles, first model how to factor with algebra tiles, and then look for connections within a generic rectangle.
   a. Using algebra tiles, factor \(2x^2 + 5x + 3\); that is, use the tiles to build a rectangle, and then write its area as a product.
      \[ \text{Product: } (2x+3)(x+1) \]
      \[ \text{Sum: } 2x^2 + 5x + 3 \]

   b. Miguel wants to use a generic rectangle to factor \(3x^2 + 10x + 8\). He knows that \(3x^2\) and \(8\) go into the rectangle in the locations shown at right. Finish the rectangle by deciding how to place the ten \(x\)-terms. Then write the area as a product.
      \[ \text{One corner should contain a } 4x \text{ and the other should be a } 6x. \text{ Product: } (3x+4)(x+2) \]

   c. Kelly wants to find a shortcut to factor \(2x^2 + 7x + 6\). She knows that \(2x^2\) and \(6\) go into the rectangle in the locations shown at right. She also remembers Casey's pattern for diagonals. Without actually factoring yet, what do you know about the missing two parts of the generic rectangle?
      Their sum is \(7x\) and the product is \(12x^2\).

   d. To complete Kelly's generic rectangle, you need two \(x\)-terms that have a sum of \(7x\) and a product of \(12x^2\). Create and solve a Diamond Problem that represents this situation.
8-15. Factoring with a generic rectangle is especially convenient when algebra tiles are not available or when the number of necessary tiles becomes too large to manage. Using a Diamond Problem helps avoid guessing and checking, which can at times be challenging. Use the process from problem 8-14 to factor $6x^2 + 17x + 12$.

The questions below will guide your process.

a. When given a trinomial, such as $6x^2 + 17x + 12$, what two parts of a generic rectangle can you quickly complete?

   One corner contains $6x^2$ and the opposite corner contains 12.

   \[ \begin{array}{c|c} 
   6x^2 & 12 \\
   \hline 
   \end{array} \]

b. How can you set up a Diamond Problem to help factor a trinomial such as $6x^2 + 17x + 12$? What goes on the top? What goes on the bottom?

   The product of $a$ and $c$ go on top, the $b$ goes on the bottom.

   \[ \begin{array}{ccc} 
   a & b \\
   \hline 
   6x^2 + 17x + 12 & \quad \\
   \end{array} \]

   \[ a = 6, b = 17, c = 12 \]

   \[ \frac{72x^2}{abc} \]

   \[ b \rightarrow 17 \times \]

c. Solve the Diamond Problem for $6x^2 + 17x + 12$ and complete its generic rectangle.

\[ \begin{array}{c|c}
 72x^2 & \\
\hline 
q x & 8x \\
\hline 
\end{array} \]

\[ \begin{array}{c|c|c}
3 & 9x & 12 \\
\hline 
2x & 6x^2 & 8x \\
\hline 
3x & 4 & \\
\end{array} \]

\[ \text{Product: } (3x+4)(2x+3) \]

d. Write the area of the rectangle as a product.

\[ \text{Product: } (3x+4)(2x+3) \]

8-16. Use the process you developed in problem 8-14 to factor the following quadratics, if possible. If a quadratic cannot be factored, justify your conclusion.

a. $x^2 + 9x + 18$

\[ \begin{array}{c|c|c|c}
18x^2 & \\
\hline 
6x & \\
\hline 
9x & \\
\hline 
\end{array} \]

\[ \begin{array}{c|c|c}
6x & 18 \\
\hline 
x^2 & 3x \\
\hline 
x & 3 \\
\hline 
\end{array} \]

\[ (x+6)(x+3) \]

b. $4x^2 + 17x - 15$

\[ \begin{array}{c|c|c|c|c}
-60x^2 & \\
\hline 
20x & \\
\hline 
-3x & \\
\hline 
\end{array} \]

\[ \begin{array}{c|c|c|c}
20x & -15 \\
\hline 
4x^2 & -3x \\
\hline 
4x & -3 \\
\hline 
\end{array} \]

\[ (x+5)(4x-3) \]

c. $4x^2 - 8x + 3$

\[ \begin{array}{c|c|c|c|c}
12x^2 & \\
\hline 
-6x & \\
\hline 
-6x & \\
\hline 
-3 & \\
\hline 
\end{array} \]

\[ \begin{array}{c|c|c|c}
-6x & 3 \\
\hline 
2x & -2x \\
\hline 
4x^2 & -2x \\
\hline 
2x & -1 \\
\hline 
\end{array} \]

\[ (2x-3)(2x-1) \]

d. $3x^2 + 5x - 3$

\[ \begin{array}{c|c|c|c|c}
-9x^2 & \\
\hline 
5x & \\
\hline 
\end{array} \]

\[ \text{Not factorable} \]

\[ 5x \]
Steps to factor using diamonds and rectangles:

Example 8-14

\[ 2x^2 + 5x + 3 \]

1. Draw a “diamond”. Then place the middle term on the bottom and the product of the first and last term on top.

\[ \begin{array}{c}
\times \\
2x^2 \\
\times \\
5x \\
\times \\
3
\end{array} \]

\[ 2x^2 \cdot 3 = 6x^2 \]

2. Solve the diamond. Hint: The numbers on the side must multiply to the number on top, and the numbers on the side must add to the number on the bottom.

\[ \begin{array}{c}
6x^2 \\
2x \\
\times \\
3 \\
\times \\
5x
\end{array} \]

\[ 6x \cdot 1x = 6x^2 \]

\[ 2x \cdot 3 \]

3. Create a rectangle. Add first term to the bottom left box; the last term to the top right box; the two side terms from the diamond to the remaining 2 boxes.

\[ \begin{array}{c}
2x \\
3 \\
\times \\
2x \\
\times \\
3
\end{array} \]

4. Check that your rectangle is correct by multiplying the diagonals. They should equal each other.

\[ 2x^2 \cdot 3 = 6x^2 \]

\[ 3x \cdot 2x = 6x^2 \]

5. Find greatest common factors to find the outer edges of the rectangle

\[ \begin{array}{c}
1 \\
2x \\
\times \\
3 \\
\times \\
2x \\
\times \\
3
\end{array} \]

6. Write the factors

\[ (x + 1)(2x + 3) \]

7. Foil to check

\[ \begin{array}{c}
0 \\
2x^2 \\
\underline{3x} \\
2x \\
\underline{+3}
\end{array} \]

\[ 2x^2 + 5x + 3 \]
Practice your new method for factoring quadratic expressions without tiles as you consider special types of quadratic expressions.

8-24. Factor each quadratic expression below, if possible. Use a Diamond Problem and generic rectangle for each one.

a. \(x^2 + 6x + 9\)

b. \(2x^2 + 5x + 3\)

c. \(x^2 + 5x - 7\)

\[\text{Not factorable}\]

d. \(3m^2 + m - 14\)

8-25. SPECIAL CASES

Most quadratic expressions are written in the form \(ax^2 + bx + c\). But what if a term is missing? Or what if the terms are in a different order? Consider these questions while you factor the expressions below. Share your ideas with your teammates and be prepared to demonstrate your process for the class.

a. \(9x^2 - 4\) \(\rightarrow\) \(9x^2 + 0x - 4\)

b. \(12x^2 - 16x\) \(\rightarrow\) \(12x^2 - 16x + 0\)

c. \(3 + 8k^2 - 10k\) \(\rightarrow\) \(8k^2 - 10k + 3\)

d. \(40 - 100m\)

\[20(3-5m)\]
8-26. Now turn your attention to the quadratic expression below. Use a generic rectangle and Diamond Problem to factor this expression. Compare your answer with your teammates’ answers. Is there more than one possible answer?

\[ 4x^2 - 10x - 6 \]

**Method 1**

\[
\begin{array}{c}
-24x^2 \\
-12x \\
-10x
\end{array}
\]

\[
\begin{array}{c|cc}
 & -12x & -6 \\
-2x & 4x^2 & 3x
\end{array}
\]

\[(2x-3)(2x+1)\]

**Method 2**

\[
\begin{array}{c}
-24x^2 \\
-12x \\
-10x
\end{array}
\]

\[
\begin{array}{c|cc}
 & -12x & -6 \\
-3 & 4x^2 & 2x
\end{array}
\]

\[(4x+2)(x-3)\]

Two different answers, both are correct.
35. Review what you have learned by factoring the following expressions, if possible.
   a. $9x^2 - 12x + 4$
   b. $81m^2 - 1$

   c. $28 + x^2 - 11x$

   d. $3n^2 + 9n + 6$

8-36. Compare your solutions for problem 8-35 with the rest of your class.
   a. Is there more than one factored form of $3n^2 + 9n + 6$? Why or why not?
   
   Yes.
   
   b. Why does $3n^2 + 9n + 6$ have more than one factored form while the other quadratics in problem 8-35 only have one possible answer? Look for clues in the original expression $(3n^2 + 9n + 6)$ and in the different factored forms.

   This has more than one factored form because it has a greatest common factor of three that can be factored out of the expression.

   ex. $3(n^2 + 3n + 2)$
c. Without factoring, predict which quadratic expressions below may have more than one factored form. Be prepared to defend your choice to the rest of the class.

i. \(12t^2 - 10t + 2\)

More than one: \(2(6t^2 - 5t + 1)\)

ii. \(5p^2 - 23p - 10\)

Only one

iii. \(10x^2 + 25x - 15\)

More than one: \(5(2x^2 + 5x - 3)\)

iv. \(3k^2 + 7k - 6\)

Only one

8-37. In part (c) of problem 8-36, you should have noticed that each term in \(12t^2 - 10t + 2\) is divisible by 2. That is, it has a common factor of 2.

a. An expression is considered completely factored if none of the factors can be factored any more. Often it is easiest to remove common factors first, before factoring with a generic rectangle. Rewrite this expression \(10x^2 + 25x - 15\) with the common factor factored out.

\[5(2x^2 + 5x - 3)\]

b. Your result in part (a) is not completely factored if either factor can be factored. Factor \(10x^2 + 25x - 15\) completely.

\[
\begin{array}{c|c|c}
-6x^2 & 6x & -x \\
6x & -1 & 5x \\
5x & & \\
\end{array}
\]

\[3(2x^2 - 5x + 3)(2x - 1)\]

\[\frac{3x^3}{5x}\]

8-38. Factor each of the following expressions as completely as possible.

a. \(5x^2 + 15x - 20\)

\[5(x^2 + 3x - 4)\]

b. \(3x^3 - 6x^2 - 45x\)

\[3x(x^2 - 2x - 15)\]
45. Your team will be assigned several of the quadratic expressions below to factor. Look for similarities and differences among the expressions and their corresponding factored forms.

a. \(x^2 - 49\)
   \[
   (x - 7)(x + 7) \\
   \text{Difference of squares}
   
   \]

b. \(x^2 + 2x - 24\)
   \[
   \begin{array}{c|c|c}
   -24x^2 & 6x & -24 \\
   \hline
   6x & -4x & \ \\
   \hline
   x^2 & -4x & \\
   \hline
   x & -4 & \\
   \end{array} \\
   (x + 6)(x - 4)
   
   \]

c. \(x^2 - 10x + 25\)
   \[
   (x - 5)^2 \\
   \text{Perfect Square}
   
   \]

d. \(9x^2 + 12x + 4\)
   \[
   (3x + 2)^2 \\
   \text{Perfect Square}
   
   \]

e. \(5x^2 - 4x - 1\)
   \[
   \begin{array}{c|c|c|c|c}
   -5x^2 & -5x & -1 \\
   \hline
   -1 & -5x & -1 \\
   \hline
   -1 & -5x & -1 \\
   \hline
   5x^2 & 1x & \\
   \hline
   5x & 1 & \\
   \end{array} \\
   (x - 1)(5x + 1)
   
   \]

f. \(4x^2 - 25\)
   \[
   (2x - 5)(2x + 5) \\
   \text{Difference of squares}
   
   \]

g. \(x^2 - 6x + 9\)
   \[
   (x - 3)^2 \\
   \text{Perfect Square}
   
   \]

h. \(x^2 - 36\)
   \[
   (x - 6)(x + 6) \\
   \text{Difference of squares}
   
   \]

i. \(7x^2 - 20x - 3\)
   \[
   \begin{array}{c|c|c|c|c}
   -7x^2 & -21x & -3 \\
   \hline
   -3 & -21x & -3 \\
   \hline
   -1x & 7x^2 & 1x \\
   \hline
   1 & -7x & 1 \\
   \end{array} \\
   (x - 3)(7x + 1)
   
   \]

j. \(4x^2 + 20x + 25\)
   \[
   (2x + 5)^2 \\
   \text{Perfect Square}
   
   \]

k. \(x^2 + 4\)
   \[
   \text{Not Factorable}
   
   \]

l. \(9x^2 - 1\)
   \[
   (3x - 1)(3x + 1) \\
   \text{Difference of squares}
   
   \]
8-46. Which of the following quadratic expressions fit the patterns you found in problem 8-45? Factor each of the following expressions using your new shortcuts, if possible.

a. $25x^2 - 1$
   \[ (5x - 1)(5x + 1) \]
   **Difference of Squares**

b. $x^2 - 5x - 36$
   \[ -36x^2 \]
   \[ -9 \]
   \[ 4x \]
   \[ -9 \]
   \[ -36 \]
   \[ 9x \]
   \[ 36 \]
   \[ x^2 \]
   \[ 4x \]
   \[ x \]
   \[ 4 \]
   \[ (x - 9)(x + 4) \]

c. $x^2 + 8x + 16$
   \[ (x + 4)^2 \]
   **Perfect Square**

d. $9x^2 - 12x + 4$
   \[ (3x - 2)^2 \]
   **Perfect Square**

e. $9x^2 + 4$
   **Not Factorable**

f. $9x^2 - 100$
   \[ (3x - 10)(3x + 10) \]
   **Difference of Squares**